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Original Research Article

Habitat complexity of a discrete predator-prey model with Hassell-Varley type functional response

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Abstract: Prey-predator models with a refuge effect are particularly significant in ecology. The two common conceptions of refuge in the literature are continual refuge and refuge pro-portionate to prey. Academics are already paying attention to new types of refuge concepts. Prey-predator interaction has become a prominent issue in recent biomathe-matical studies due to its environmental influence. In this paper, the habitat complexity of a predator-prey model with Hassell-Varley type functional response is considered. For this, we focused our study on the question of existence and uniqueness in Sec. 2. And Sec. 3 is devoted to show a generalized stability. Note that this representation also al-lows us to generalize the results obtained recently in the literature. In Sec. 4, we have studied the numerical algorithm for the suggested problem. The paper is ended by two examples illustrating our results.

Keywords: Predator-prey model, Hassell-Varley type functional response.

1. Introduction of the model

In ecology, prey-predator models featuring a refuge effect are extremely important. In the available literature, the most prominent notions of refuge are constant refuge and refuge proportionate to prey. New forms of refuge concepts are already drawing academics' attention. Because of its impact on the environment, prey-predator interaction has become a hot topic in the contemporary biomathematical studies. Many researchers have worked in investigatingvarious aspects of the dynamical behaviour of this subject matter in ecology, as well as the accompanying growth of population models. Some prey populations benefit from natural protection in the form of refuge dimensions. In other cases, several aspects allow for a longer prey-predator contact, lowering the risk of extinction owing to predation. Many scholars in the discrete area of waste concepts have studied this phenomenon extensively. We refer for reference to the work in[1–5].

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The nature of the birth process is a continuous matter in populations with overlapping gen-erations; thus, the predator-prey interaction is usually developed using deterministic models such as ordinary differential equations. Several species, such as monocarpic plants and semel-parous animals, exhibit discrete nonoverlapping generation characters as well as predictable birth and breeding seasons. Difference equations or other forms, such as discrete-time map-pings, are used to represent their interactions. The dynamical behaviour of a discrete-time prey-predator model is typically more sophisticated than that of the equivalent continuous-time models[6–9].

Different mathematical tools were used for the considerations of the predator prey models. For example, in[10] the Allee effect and the fear effect are proven in a predator-prey paradigm. For the boundedness, they applied the comparison principle. The model's equilibrium point and nonzero equilibrium point were studied, as well as the local stability conditions were examined. The cross-sectional conditions of transcritical bifurcation and Hopf bifurcation were also determined using the Sotomayor theorem. While in[11] the authors proposed and analysed an age-structure predator prey dynamical system in which predators were classified into juvenile and mature predators utilising a MonodHaldane-type response function. The stability and bifurcation of the system were explored both analytically and quantitatively. By creating a suitable Lyapunov function, we studied global stability around the interior equilibrium point E^*. Furthermore, in[12] the authors investigated the role of diffusion and nonlocal prey eating on the population distributions of an interacting generalist predator and its focused prey species. For this, they first used linear analysis to derive the criteria that lead to Turing instability. And with the usage of a weakly nonlinear analysis, they obtained acubic Stuart-Landau equation that governs the amplitude of the generated patterns near the Turing bifurcation boundary. In[13], the authors studied fear, refuge, and harvesting elements are in a predatorprey model with infection in the prey population and evaluated an expression for the fundamental reproduction number. The global stability of disease-free and endemic equilibria was determined based on reproduction number. They also discovered that by regulating the value of the basic reproduction number less than one, sickness is eliminated from the system. In addition, transcritical and Hopf bifurcations were investigated for the deterministic model. In[14] predator-prey system with a Holling type function response and prey refuge was provided. The dynamical behaviour of the considered system was investi-gated by the help of analytical methodologies, including stability, limit cycle, and bifurcation. The findings reveal that the shape of the functional response has a significant impact on the system's dynamics. The intriguing conclusion is that the prey refuge has a destabilising effect in some circumstances. In[15] the researchers explored the effects of predator harvesting in a two-dimensional preypredator model with Holling type III functional response. Their major goal was to investigate how the dynamical behaviour of the preypredator model changes when non-linear predator harvesting is present. As the parameters are changed, the model system displays complicated dynamics. Under a specific parametric condition with non-negative besof system parameters, the presence and stability criterion of several equilibrium points w asexamined. Under various parametric conditions, the system exhibits saddle-node bifurcati on,transcritical bifurcation, Hopf bifurcation, and Bogdanov-Takens bifurcation. For more de tail,we refer the readers to [16–22].

In [33], Hassell and Varley introduced a general predator-prey system (PPS), in whicht he functional response dependents on the predator density in different way. It is calledaH assell–Varley (HV) type functional response which takes the following form

$$\begin{aligned} \frac{du_1}{dx} &= u_1\left[r_1\left(1-\frac{u_1}{\xi(1-\eta)}\right)-\frac{\zeta(1-a_1)u_2}{u_1+a_2\zeta(1-a_1){u_2}^{\sigma_{HV}}}\right], \\ \frac{du_2}{dx} &= u_2\left[\frac{\hbar\zeta(1-a_1)u_1}{u_1+a_2\zeta(1-a_1){u_2}^{\sigma_{HV}}}-r_2\right], \end{aligned} \tag{1.1}$$

where $\sigma_{HV} \in \jmath := (0,1)$ is called the HV constant (detail description of the parameter can be seen in [33]). For terrestrial predators that form a fixed number of tight groups, it is oftenr easonable to assume $\sigma_{HV} = \frac{1}{2}$. For aquatic predators that form a fixed number of tights groups, $\sigma_{HV} = \frac{1}{3}$ may be more appropriate. It is worth pointing out that during the course of the predator-prey interaction when predators do not form groups, one can assume that the HV constant is equal to 1, that is, $\sigma_{HV} = 1$. For better description, the complex problem of the interaction between prey and predator has been considered via discrete models, the reader may refer to [31–36] for more details.

The aim of this work is to formulate a discrete prey-predator model based on the same assumptions as (1.1). There are different reasons for using discrete mathematical models. While they are often preferred due to their computational convenience, they are also more appropriate for modelling non-overlapping generations, Moreover, using discrete-time models is more efficient for computation and numerical simulations. By analysis it is proved that the discrete-time model has different properties and structures compared with the continuous one.

In this paper, we consider the following discrete-time predator-prey system:

$$\begin{aligned} (u_1)_{n+1} &= (u_1)_n\left[r_1\left(1-\frac{(u_1)_n}{\xi(1-\eta)}\right)-\frac{\zeta(1-a_1)(u_2)_n}{(u_1)_n+a_2\zeta(1-a_1)(u_2)_n^{\sigma_{HV}}}\right], \\ (u_2)_{n+1} &= (u_2)_n\left[\frac{\hbar\zeta(1-a_1)(u_1)_n}{(u_1)_n+a_2\zeta(1-a_1)(u_2)_n^{\sigma_{HV}}}-r_2\right], \end{aligned} \tag{1.2}$$

assume $\sigma_{HV} = 1$.

Here we focused our study on the question of existence and uniqueness in Sec. 2. And Sec. 3 is devoted to show a generalized stability. Note that this representation also allows us to generalize the results obtained recently in the literature. In Sec. 4, we have studied the numerical algorithm for the suggested problem. The paper is ended by two examples illustrating our results. The work is summarised in 5. In future, we aim for reconsideration of the model in fractional version, with the help of the work in [22–30].

2 The dynamic behavior of the system

2.1 The fixed points and their feasibility

The fixed pointscan be obtained by solving the following nonlinear system

$$\begin{aligned} u_1 &= u_1\left[r_1\left(1-\frac{u_1}{\xi(1-\eta)}\right)-\frac{\zeta(1-a_1)u_2}{u_1+a_2\zeta(1-a_1)u_2}\right],\\ u_2 &= u_2\left[\frac{\hbar\zeta(1-a_1)u_1}{u_1+a_2\zeta(1-a_1)u_2}-r_2\right]. \end{aligned} \tag{2.1}$$

Solving the system (2.1) we get the fixed points like:

(i) The trivial fixed point $\mathfrak{E}_0(0,0)$, which is always feasible.

(ii) The predator free fixed point $\mathfrak{E}_1(\xi(1-\eta),0)$, which is feasible if $0<\eta<1$.

(iii) The coexistence fixed point $\mathfrak{E}_2(u_1^*,u_2^*)$, where

$$\begin{aligned} u_1^* &= \xi(1-\eta)\left\{1+\frac{1}{a_2r_1}\left(\frac{r_2}{\hbar\zeta(1-a_1)}-1\right)\right\}\\ u_2^* &= \frac{\xi(1-\eta)(\hbar\zeta(1-a_1)-r_2)}{a_2(1-a_1)r_2\zeta}\left\{1-\frac{\hbar\zeta(1-a_1)-r_2}{a_2\hbar\zeta r_1(1-a_1)}\right\}, \end{aligned} \tag{2.2}$$

which is feasible if $0<a_1<1-\frac{r_2}{\hbar\zeta(1-a_2r_1)}$ and $0<r_2<\hbar\zeta(1-a_1)$.

2.2 Stability analysis

In this section we shall discuss the local stability analysis of the system (1.2) for each of the fixed points. For system (1.2) the Jacobian matrix is as follows

$$J(u_1,u_2)=\begin{bmatrix} r_1\left\{1-\frac{2u_1}{(1-\eta)\xi}\right\}-\frac{a_2\zeta^2(1-a_1)^2u_2^2}{[u_1+a_2\zeta(1-a_1)u_2]^2} & -\frac{\zeta(1-a_1)u_1^2}{[u_1+a_2\zeta(1-a_1)u_2]^2}\\ \frac{a_2\hbar\zeta^2(1-a_1)^2u_2^2}{[u_1+a_2\zeta(1-a_1)u_2]^2} & \frac{\hbar u(1-a_1)u_1^2}{[u_1+a_2\zeta(1-a_1)u_2]^2}-r_2 \end{bmatrix}. \tag{2.3}$$

For the Jacobian matrix (2.3), we have the following characteristic equation

$$|\lambda I-J(u_1,u_2)|=0,$$

hence

$$\lambda^2-\mathrm{Trace}(J)\lambda+\mathrm{Det}(J)=0. \tag{2.4}$$

Here λ is the eigenvalue of $J(\mathrm{u}_1, \mathrm{u}_2)$ and Trace(J) and Det(J) are the trace and determinant of the Jacobian matrix $J(\mathrm{u}_1, \mathrm{u}_2)$ respectively, where

$$\text{Trace}\left(J(\mathrm{u}_1, \mathrm{u}_2)\right) = r_1\left\{1 - \frac{2\mathrm{u}_1}{\xi(1-\eta)}\right\} - \frac{a_2\zeta^2(1-a_1)^2\mathrm{u}_2^2}{[\mathrm{u}_1 + a_2\zeta(1-a_1)\mathrm{u}_2]^2}$$

$$+ \frac{\hbar\zeta(1-a_1)\mathrm{u}_1^2}{[\mathrm{u}_1 + a_2\zeta(1-a_1)\mathrm{u}_2]^2} - r_2,$$

$$\text{Det}\left(J(\mathrm{u}_1, \mathrm{u}_2)\right) = r_1\left\{1 - \frac{2\mathrm{u}_1}{\xi(1-\eta)}\right\}\left\{\frac{\hbar\zeta(1-a_1)\mathrm{u}_1^2}{[\mathrm{u}_1 + a_2\zeta(1-a_1)\mathrm{u}_2]^2} - r_2\right\}$$

$$+ \frac{r_2 a_2\zeta^2(1-a_1)^2\mathrm{u}_2^2}{[\mathrm{u}_1 + a_2\zeta(1-a_1)\mathrm{u}_2]^2}.$$

Therefore, depending upon the value of Det(J) we can classified the system (1.2), i.e., if

i) Det$(J)< 1$, then the dynamical system is called dissipative.

ii) Det$(J)= 1$, then the dynamical system is called conservative.

iii) Otherwise, the dynamical system is called undissipated.

In order to stability analysis of the fixed points of the system (1.2), first we give the following lemma that can be easily proved by the relation between roots and coefficients of the characteristic Eq. (2.4) of the system (1.2).

Lemma 2.1. *Let* $\mathbb{G}(\lambda) = \lambda^2 - \alpha\lambda + \beta$. *Suppose that* $\mathbb{G}(1) > 0$*;* λ_1, λ_2 *are the two roots of* $\mathbb{G}(\lambda) = 0$. *Then*

i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *iff* $\mathbb{G}(-1) > 0$ *and* $\beta < 1$.

ii) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *iff* $\mathbb{G}(-1) < 0$.

iii) $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *iff* $\mathbb{G}(-1) > 0$ *and* $\beta > 1$.

iv) $\lambda_1 = -1$ *and* $\lambda_2 \neq 1$ *iff* $\mathbb{G}(-1) = 0$ *and* $\alpha \neq 0, 2$.

v) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2|$ *iff* $\alpha^2 - 4\beta < 0$ *and* $\beta = 1$.

Lemma 2.2. *Let* λ_1, λ_2 *are two roots of (2.4), are called the eigenvalues of fixed point* $(\bar{\mathrm{u}}_1, \bar{\mathrm{u}}_2)$ *then:*

i) A fixed point $(\bar{\mathrm{u}}_1, \bar{\mathrm{u}}_2)$ *is called sink if* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$, *i.e., the sink is locally asymptotically stable.*

ii) A fixed point $(\bar{\mathrm{u}}_1, \bar{\mathrm{u}}_2)$ *is called source if* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$, *i.e., the source is locally unstable.*

iii) A fixed point $(\bar{u}_1, \bar{u}_2)$ *is called saddle if* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *or* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$.

iv) A fixed point $(\bar{u}_1, \bar{u}_2)$ *is called non-hyperbolic if either* $|\lambda_1| = 1$ *or* $|\lambda_2| = 1$.

Theorem 2.3. *The fixed point* $\mathfrak{E}_0(0,0)$ *is a (i) sink if* $r_j < 1$*; (ii) source if* $r_j > 1$*; (iii) non-hyperbolic if* $r_j = 1$*; (iv) saddle for other values of* r_j *here* $j = 1,2$.

Proof. The Jacobian matrix for the the fixed point E_0 becomes

$$J(\mathfrak{E}_0) = \begin{pmatrix} r_1 & 0 \\ 0 & -r_2 \end{pmatrix},$$

and the associated eigenvalues are $\lambda_1 = r_1$, $\lambda_2 = -r_2$. Now following the lemma 2.2, we can say that $\mathfrak{E}_0$ is a sink, source, non-hyperbolic if $r_i < 1$, $r_i > 1$ and $r_i = 1$ holds respectively, $i = 1,2$. For other values of r_i, $\mathfrak{E}_0$ is called a saddle point. □

Theorem 2.4. *The fixed point* $\mathfrak{E}_1\,(\xi(1-\eta), 0)$ *is a i) sink if* $r_1 < 1$,

$$1 - \frac{1+r_2}{\hbar\zeta} < a_1 < 1 + \frac{1-r_2}{\hbar\zeta};$$

ii) source if $r_1 > 1$,

$$1 - \frac{1+r_2}{\hbar\zeta} > a_1 > 1 + \frac{1-a_2}{\hbar\zeta};$$

iii) non-hyperbolic if $r_1 = 1$, $1 - \frac{1+r_2}{\hbar\zeta} = a_1$ *or* $a_1 = 1 + \frac{1-r_2}{\hbar\zeta}$*; and iv) saddle for other values of parameters.*

Proof. The Jacobian matrix for the the fixed point $\mathfrak{E}_0$ becomes

$$J(\mathfrak{E}_1) = \begin{bmatrix} -r_1 & -\zeta(1-a_1) \\ 0 & -r_2 + \hbar\zeta(1-a_1) \end{bmatrix},$$

and the associated eigenvalues are $\lambda_1 = -r_1$, $\lambda_2 = -r_2 + \hbar\zeta(1-a_1)$. Now following the lemma 2.2, we can say that $\mathfrak{E}_1$ is a sink, source, non-hyperbolic if $r_1 < 1$,

$$1 - \frac{1+r_2}{\hbar\zeta} < a_1 < 1 + \frac{1-r_2}{\hbar\zeta},$$

$r_1 > 1$,

$$1 - \frac{1+r_2}{\hbar\zeta} > a_1 > 1 + \frac{1-r_2}{\hbar\zeta}$$

and $r_1 = 1$, $1 - \frac{1+r_2}{\hbar\zeta} = a_1$ or $a_1 = 1 + \frac{1-r_2}{\hbar\zeta}$ holds respectively. For other values of parameters $\mathfrak{E}_1$ is called a saddle point.

Now, from condition (iii) of theorem 2.4, for $r_1 = 1$, we have $\lambda_1 = -1$ and $\lambda_2 \neq -1, 1$. Therefore, let us define the sets $A_{\mathfrak{E}_1}$ and $B_{\mathfrak{E}_1}$, where all the parameters located, as:

$$\mathcal{A}_{\mathfrak{E}_1} = \Big\{(r_1, \xi, \zeta, \eta, a_1, a_2, r_2, \hbar) : 0 < r_1 = 1, a_1 > 1 - \frac{1 + r_2}{\hbar\zeta}, \\ a_1 \neq 1 + \frac{1 - r_2}{\hbar\zeta}; 0 < a_1 < 1\Big\},$$

$$\mathcal{B}_{\mathfrak{E}_1} = \Big\{(r_1, \xi, \zeta, \eta, a_1, a_2, r_2, \hbar) : a_1 = 1 - \frac{1 + r_2}{\hbar\zeta}, r_1 \neq 1, r_1 > 1; 0 < a_1 < 1\Big\}.$$

Now if the parameters moving in the very small neighborhood of $\mathcal{A}_{\mathfrak{E}_1}$ and $\mathcal{B}_{\mathfrak{E}_1}$, then the fixed point $\mathfrak{E}_1$ will passes through a flip bifurcation. Also, there is a center manifold $u_2 = 0$ of system (1.2) at $\mathfrak{E}_1$ [?]. Hence in this case the predator becomes extinct and the prey undergoes the period doubling bifurcation to chaos in the sense of Li-York by choosing bifurcation parameter 'r_1'.

Theorem 2.5. *If* $0 < a_1 < 1 - \frac{r_2}{\hbar\zeta(1-a_2 r_1)}$ *and* $0 < r_2 < \hbar\zeta(1 - a_1)$, *then the coexistence fixed point* $\mathfrak{E}_2(u_1^*, u_2^*)$ *is called*

(i) sink if, $a_1 > 1 - \frac{2r_2}{\hbar\zeta(1-a_2 r_1)}$ *and* $0 < r_2 < \hbar\zeta(1 - a_1)$;

(ii) source if, $0 < a_1 < 1 - \frac{2r_2}{\hbar\zeta(1-a_2 r_1)}$ *and* $0 < r_1 < \hbar\zeta(1 - a_1)$;

(iii) non-hyperbolic if, $a_1 = 1 - \frac{2r_2}{\hbar\zeta(1-a_2 r_1)}$ *and* $0 < r_2 < \hbar\zeta(1 - a_1)$.

Proof. The expression for Trace(J) and Det(J) for the Jacobian matrix (2.3) has been given after Eq. (2.4). After computing the value of Trace($J(\mathfrak{E}_2)$) and Det($J(\mathfrak{E}_2)$) we may get the following conditions

I) 1+Trace($J(\mathfrak{E}_2)$)+Det($J(\mathfrak{E}_2)$)> 0,

II) 1−Trace($J(\mathfrak{E}_2)$)+Det($J(\mathfrak{E}_2)$)> 0,

III) 1−Det($J(\mathfrak{E}_2)$)> 0.

If all the above conditions holds simultaneously for the fixed point $\mathfrak{E}_2$, we can say that it is stable. After some computations we'll get the results (i)-(iii). Now, from condition (iii) of theorem 2.5 and from Lemma 2.1, we can see that one of the eigenvalues of the positive fixed point $\mathfrak{E}_2$ is −1 and the other is neither 1 nor −1. Therefore, let us define the sets $\bar{\mathcal{A}}_{\mathfrak{E}_2}$, where all the parameters located, as

$$\bar{\mathcal{A}}_{\mathfrak{E}_2} = \Big\{(r_1, \xi, \zeta, \eta, a_1, a_2, r_2, \hbar, \sigma) : a_1 = 1 - \frac{2r_2}{\mathfrak{E}(1 - a_2 r_1)}, \\ 0 < r_2 < \hbar\zeta(1 - a_1); 0 < a_1 < 1\Big\}.$$

3 Bifurcation

3.1 Flip bifurcation

In this section, based on the previous analysis, we discuss the flip bifurcation of the predator free fixed point $\mathfrak{E}_1$. We choose parameter a_1 as a bifurcation parameter to study the flip bifurcation at the equilibrium point by using center manifold theorem and bifurcation theory in [31, 32]. We first discuss the flip bifurcation of system (2.1) at $\mathfrak{E}_1$ when parametersvar y in the small neighborhood of $\bar{\mathcal{A}}_{\mathfrak{E}_1}$. Taking parameters $(\acute{r}_1, \acute{\xi}, \acute{\zeta}, \acute{\eta}, \acute{a}_1, \acute{a}_2, \acute{r}_2, \acute{\hbar})$, arbitrar-ily from $\bar{\mathcal{A}}_{\mathfrak{E}_1}$, for simplicity we consider system (2.1) with $(r_1, \xi, \zeta, \eta, a_1, a_2, r_2, \hbar)$ in place of $(\acute{r}_1, \acute{\xi}, \acute{\zeta}, \acute{\eta}, \acute{a}_1, \acute{a}_2, \acute{r}_2, \acute{\hbar})$, which is described by

$$\begin{aligned} u_1 &= u_1\left[r_1\left(1 - \frac{u_1}{\xi(1-\eta)}\right) - \frac{\zeta(1-a_1)u_2}{u_1 + a_2\zeta(1-a_1)u_2}\right], \\ u_2 &= u_2\left[\frac{\hbar\zeta(1-a_1)u_1}{u_1 + a_2\zeta(1-a_1)u_2} - r_2\right]. \end{aligned} \tag{3.1}$$

Giving a perturbation $\bar{r}_1$ on parameter r_1, we consider a perturbation of model (3.1) as follows

$$\begin{aligned} u_1 &= u_1\left[(\bar{r}_1 + r_1)\left(1 - \frac{u_1}{\xi(1-\eta)}\right) - \frac{\zeta(1-a_1)u_2}{u_1 + a_2\zeta(1-a_1)u_2}\right], \\ u_2 &= u_2\left[\frac{\hbar\zeta(1-a_1)u_1}{u_1 + a_2\zeta(1-a_1)u_2} - r_2\right]. \end{aligned} \tag{3.2}$$

where, $|\bar{r}_1| << 1$. Let $v_1 = u_1 - \acute{u}_1$ and $v_2 = u_2 - \acute{u}_2$. Then we transform the fixed point $\mathfrak{E}_1$ of system (3.2) into origin. We have

$$\begin{aligned} u_1 &= (v_1 + \acute{u}_1)\left[(r_1 + \bar{r}_1)\left(1 - \frac{(v_1 + \acute{u}_1)}{\xi(1-\eta)}\right) - \frac{\zeta(1-a_1)(v_2 + \acute{u}_2)}{(v_1 + \acute{u}_1) + a_2\zeta(1-a_1)(v_2 + \acute{u}_2)}\right], \\ u_2 &= (v_2 + \acute{u}_2)\left[\frac{\hbar\zeta(1-a_1)(v_1 + \acute{u}_1)}{(v_1 + \acute{u}_1) + a_2\zeta(1-a_1)(v_2 + \acute{u}_2)} - r_2\right]. \end{aligned} \tag{3.3}$$

Expanding system (3.3) as Taylor series at $(v_1, v_2, \bar{r}_1) = (0, 0, 0)$ up to second order, then it becomes the following model

$$\begin{aligned} v_1 &= \mathfrak{d}_{11}v_1 + \mathfrak{d}_{12}v_1 + \mathfrak{d}_{13}v_1^2 + \mathfrak{d}_{14}v_1v_2 + \mathfrak{e}_{12}v_1\bar{r}_1 + \mathfrak{e}_{13}v_1^2\bar{r}_1 + \mathcal{O}(v_1, v_2)^3, \\ v_2 &= \mathfrak{d}_{21}v_1 + \mathfrak{d}_{22}v_2 + \mathfrak{d}_{23}v_1^2 + \mathfrak{d}_{24}v_1v_2 + \mathcal{O}(v_1, v_2)^3, \end{aligned} \tag{3.4}$$

where, $\mathfrak{d}_{11} = r_1 - \frac{2r_1}{\xi(1-\eta)}\acute{u}_1$, $\mathfrak{d}_{12} = \zeta(1-a_1)$, $\mathfrak{d}_{13} = -r_1$,

$$\mathfrak{d}_{14} = \frac{\zeta(1-a_1)}{\acute{u}_1} - \frac{\zeta(1-a_1)^2}{\acute{u}_1} + \frac{u_1(1-a_1)}{\acute{u}_1},$$

$\mathfrak{e}_{12} = 1 - 2\acute{u}_1$, $\mathfrak{e}_{13} = -1 + \eta$, $\mathfrak{d}_{21} = 2\xi(1-\eta)$, $\mathfrak{d}_{22} = -r_2 + \hbar\zeta(1-a_1)$, $\mathfrak{d}_{23} = -2r_2$ and

$$\mathfrak{d}_{24} = \frac{\zeta(1-a_1)}{\acute{u}_1} + \frac{\zeta(1-a_1)^2}{\acute{u}_1} - \frac{a_1\zeta^2(1-a_1)}{\acute{u}_1}.$$

We construct an invertible matrix as

$$\Upsilon = \begin{pmatrix} \mathfrak{d}_{12} & \mathfrak{d}_{12} \\ -1-\mathfrak{d}_{11} & \lambda_1 - \mathfrak{d}_{11} \end{pmatrix}$$

and use the transformation

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \Upsilon \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The model (3.4) becomes into the following form

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \varrho_1(v_1, v_2, \bar{r}_1) \\ \varrho_2(v_1, v_2, \bar{r}_1) \end{pmatrix},$$

where

$$\varrho_1(v_1, v_2, \bar{r}_1) = \frac{1}{\mathrm{Det}(\Upsilon)}\left\{(\lambda_1 - \mathfrak{d}_{11})\mathfrak{d}_{13}v_1^2 + (\lambda_1 - \mathfrak{d}_{11})\mathfrak{d}_{14}v_1v_2 + \mathcal{O}(|v_1|, |v_2|, |\bar{r}_1|)\right\},$$

$$\varrho_2(v_1, v_2, \bar{r}_1) = \frac{1}{\mathrm{Det}(\Upsilon)}\left\{(1 + \mathfrak{d}_{11})\mathfrak{d}_{13}v_1^2 + [(1 + \mathfrak{d}_{11})\mathfrak{d}_{14} - \mathfrak{d}_{12}\mathfrak{d}_{24}]v_1v_2 + \mathcal{O}(|v_1|, |v_2|, |\bar{r}_1|)\right\}.$$

Now, we determine the center manifold $\Xi^c(0,0)$ of (3.4) at fixed point $(0,0)$ in small neighborhood of $\bar{r}_1 = 0$. By center manifold theorem, we can obtain the approximate representation of the center manifold $\Xi^c(0,0)$ as follows

$$\Xi^c(0,0) = \left\{(u_1, u_2) : u_2 = \tau_0\bar{r}_1 + \tau_1u_1^2 + \tau_2u_1\bar{r}_1 + \tau_3\bar{r}_1^2 + \mathcal{O}(|u_1|, |\bar{r}_1|)^3\right\},$$

where $\mathcal{O}(|u_1|, |\bar{r}_1|)^3$ is a function of $(u_1, \bar{r}_1)$, at least of the third order and $\tau_0 = \tau_2 = \tau_3 = 0$,

$$\tau_1 = \frac{1}{\mathrm{Det}(\Upsilon)(\lambda_1 - 2)}\left\{(1 + \mathfrak{d}_{11})\mathfrak{d}_{13}\mathfrak{d}_{12}^2 + [(1 + \mathfrak{d}_{11})\mathfrak{d}_{14} + \mathfrak{d}_{12}\mathfrak{d}_{24}]\right\}.$$

We write now v_1, v_2 in terms of u_1, u_2, $\bar{r}_1$ as $v_1 = \mathfrak{d}_{12}(u_1 + \tau_1u_1^2)$,

$$v_2 = -(1 + \mathfrak{d}_{11})u_1 + (\lambda_1 - u_{11})\tau_1u_1^2.$$

Therefore the map $\mathbb{G}^*$ which is restricted to the center manifold $\Xi^c(0,0)$ is

$$\begin{aligned}\mathbb{G}^*(u_1) &= u_1 + \varrho_1(v_1, v_2, \bar{r}_1) \\ &= u_1 + \iota_0\bar{r}_1 + \iota_1u_1^2 + \iota_2u_1\bar{r}_1 + \iota_3\bar{r}_1^2 + \iota_4u_1^2 + \bar{r}_1 \\ &\quad + \iota_5u_1\bar{r}_1^2 + \iota_6u_1^3 + \iota_7\bar{r}_1^2 + \mathcal{O}(|u_1| + |a_1^*|)^3,\end{aligned} \tag{3.5}$$

where $\iota_0 = \iota_3 = \iota_5 = \iota_7 = 0$,

$$\iota_1 = \frac{1}{\mathrm{Det}(\Upsilon)}\Big[\mathfrak{d}_{12}^2(\lambda_1 - \mathfrak{d}_{11})\mathfrak{d}_{13} - \mathfrak{d}_{12}(1 + \mathfrak{d}_{11})\left\{(\lambda_1 - \mathfrak{d}_{11})\mathfrak{d}_{14} - \mathfrak{d}_{12}\mathfrak{d}_{24}\right\},$$

$\iota_2 = \frac{1}{\mathrm{Det}(\Upsilon)}\mathfrak{d}_{12}\mathfrak{e}_{12}(\lambda_1 - \mathfrak{d}_{11}),$

$$\iota_4 = \frac{1}{\mathrm{Det}(\Upsilon)}\Big[\mathfrak{d}_{12}^2\mathfrak{d}_{13}(\lambda_1 - \mathfrak{d}_{11}) - \mathfrak{d}_{12}(1+\mathfrak{d}_{11})\{(\lambda_1 - \mathfrak{d}_{11})\mathfrak{d}_{14} - \mathfrak{d}_{12}\mathfrak{d}_{24}\}$$
$$+ \mathfrak{d}_{11}\mathfrak{d}_{12}\mathfrak{e}_{12}(1+\mathfrak{d}_{11})^2(\lambda_1 - \mathfrak{d}_{11})\Big],$$
$$\iota_6 = \frac{1}{\mathrm{Det}(\Upsilon)}\Big[2\mathfrak{d}_{11}\mathfrak{d}_{12}^2\mathfrak{d}_{13}(\lambda_1 - \mathfrak{d}_{11}) + \mathfrak{d}_{11}\mathfrak{d}_{12}(\lambda_1 - 1 - 2\mathfrak{d}_{11})\{(\lambda_1 - \mathfrak{d}_{11})\mathfrak{d}_{14} - \mathfrak{d}_{12}\mathfrak{d}_{24}\}\Big].$$

In order to undergo a flip bifurcation for (3.5), we require that two discriminatory quantities m_1 and m_2 are not zero, where

$$\ell_1 = \left(\frac{\partial^2\mathbb{G}}{\partial \mathrm{u}_1\partial\bar{r}_1} + \frac{1}{2}\frac{\partial\mathbb{G}}{\partial\bar{r}_1}\times\frac{\partial^2\mathbb{G}}{\partial\mathrm{u}_1^2}\right)\Bigg|_{(0,0)} = \iota_0\iota_1 + \iota_2 \neq 0,$$

$$\ell_2 = \left(\frac{1}{6}\frac{\partial^2\mathbb{G}}{\partial\mathrm{u}_1^2} + \frac{1}{4}\left(\frac{\partial\mathbb{G}}{\partial\mathrm{u}_1}\right)^2\right)\Bigg|_{(0,0)} = \iota_6 + \iota_1^2.$$

Thus from the above analysis and the theorem [32], we obtain the following result.

Theorem 3.1. *If $\ell_1 \neq 0$, then the model (3.1) undergoes a flip bifurcation at the predator free equilibrium point $\mathfrak{E}_1$ when the parameter a_1 varies in the small neighborhood of the origin. Moreover, if $\ell_1 > 0 (< 0)$, then the period 2 points that bifurcate from $\mathfrak{E}_1$ are stable (unstable).*

3.2 Hopf-bifurcation

In this section, we shall discuss the Hopf-bifurcation of the positive fixed point $\mathfrak{E}^*(\mathrm{u}_1^*, \mathrm{u}_2^*)$ if parameters vary in the small neighborhood of $\bar{\mathcal{A}}_{\mathfrak{E}^*}$. We choose the parameter 'a_1' as a bifurcation parameter to study the Hopf-bifurcation of $\mathfrak{E}^*(\mathrm{u}_1^*, \mathrm{u}_2^*)$ by using center manifold theorem and bifurcation theory [31, 32], when parameters vary in the small neighborhood of $\bar{\mathcal{A}}_{\mathfrak{E}^*}$. Our system is

$$\begin{aligned} \mathrm{u}_1' &= \mathrm{u}_1\left[r_1\left(1 - \frac{\mathrm{u}_1}{\xi(1-\eta)}\right) - \frac{\zeta(1-a_1)\mathrm{u}_2}{\mathrm{u}_1 + a_2\zeta(1-a_1)\mathrm{u}_2}\right], \\ \mathrm{u}_2' &= \mathrm{u}_2\left[\frac{\hbar\zeta(1-a_1)\mathrm{u}_1}{\mathrm{u}_1 + a_2\zeta(1-a_1)\mathrm{u}_2} - r_2\right]. \end{aligned} \tag{3.6}$$

Now the characteristic equation for the system (3.6) at $\mathfrak{E}^*(\mathrm{u}_1^*, \mathrm{u}_2^*)$ is given by

$$\lambda^2 - \mathrm{Tr}(J)\lambda + \mathrm{Det}(J) = 0,$$

where,

$$\begin{aligned}\mathrm{Tr}(J(\mathfrak{E}^*)) = r_1\left\{1 - \frac{2\mathrm{u}_1^*}{\xi(1-\eta)}\right\} - \frac{a_2\zeta^2(1-a_1)^2\mathrm{u}_2^{*^2}}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2}\\ + \frac{\hbar\zeta(1-a_1)\mathrm{u}_1^{*^2}}{(\mathrm{u}^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} - r_2,\end{aligned}$$

$$\begin{aligned}\mathrm{Det}(J(\mathfrak{E}^*)) = r_1\left\{1 - \frac{2\mathrm{u}_1^*}{\xi(1-\eta)}\right\}\left\{\frac{\hbar\zeta(1-a_1)\mathrm{u}_1^{*^2}}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} - r_2\right\}\\ + \frac{r_2 a_2\zeta^2(1-a_1)^2\mathrm{u}_2 y^{*^2}}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2}.\end{aligned}$$

The eigenvalues of $\mathfrak{E}^*$ are a pair of complex conjugate numbers λ and $\bar{\lambda}$ with modulus 1, where,

$$\lambda = \frac{\mathrm{Tr}(J) + \sqrt{\mathrm{Tr}(J)^2 - 4\mathrm{Det}(J)}}{2}.$$

Giving a perturbation a_1^* of parameter a_1, we consider a perturbation of model (3.6) as follows

$$\begin{aligned}\mathrm{u}_1 &= \mathrm{u}_1\left[r_1\left(1 - \frac{\mathrm{u}_1}{\xi(1-\eta)}\right) - \frac{\zeta(1-(a_1+a_1^*))\mathrm{u}_2}{\mathrm{u}_1 + a_2\zeta(1-a_1)\mathrm{u}_2}\right],\\ \mathrm{u}_2 &= \mathrm{u}_2\left[\frac{\hbar\zeta(1-(a_1+a_1^*))\mathrm{u}_1}{\mathrm{u}_1 + a_2\zeta(1-a_1)\mathrm{u}_2} - r_2\right].\end{aligned} \tag{3.7}$$

where $|a_1^*| << 1$. Let $\mathrm{v}_1 = \mathrm{u}_1 - \mathrm{u}_1^*$ and $\mathrm{v}_2 = \mathrm{u}_2 - \mathrm{u}_2^*$. Then we transform the fixed point $\mathfrak{E}^*(\mathrm{u}_1^*, \mathrm{u}_2^*)$ of system (3.7) into origin. We have

$$\begin{aligned}\mathrm{v}_1 &= (\mathrm{v}_1 + \mathrm{u}_1^*)\left[r_1\left(1 - \frac{\mathrm{v}_1 + \mathrm{u}_1^*}{\xi(1-\eta)}\right) - \frac{\zeta(1-(a_1+a_1^*))(\mathrm{v}_2) + \mathrm{u}_2^*}{(\mathrm{v}_1 + \mathrm{u}_1^*) + a_2\zeta(1-a_1)(\mathrm{v}_2 + \mathrm{u}_2^*)}\right],\\ \mathrm{v}_2 &= (\mathrm{v}_2 + \mathrm{u}_2^*)\left[\frac{\hbar\zeta(1-(a_1+a_1^*))(\mathrm{v}_1 + \mathrm{u}_1^*)}{(\mathrm{v}_1 + \mathrm{u}_1^*) + a_2\zeta(1-a_1)(\mathrm{v}_2 + \mathrm{u}_2^*)} - r_2\right].\end{aligned} \tag{3.8}$$

Expanding system (3.8) as taylor series at $(\mathrm{v}_1, \mathrm{v}_2, a_1^*) = (0,0,0)$ to second order, then it becomes the following model

$$\begin{aligned}\mathrm{v}_1 &= \acute{\mathfrak{d}}_{11}\mathrm{v}_1 + \acute{\mathfrak{d}}_{12}\mathrm{v}_2 + \acute{\mathfrak{d}}_{13}\mathrm{v}_1^2 + \acute{\mathfrak{d}}_{14}\mathrm{v}_1\mathrm{v}_2 + \acute{\mathfrak{e}}_{11}\mathrm{v}_1\mathrm{v}_2 a_1^* + \acute{\mathfrak{e}}_{12}a_1^*\mathrm{v}_1\\ &\quad + \acute{\mathfrak{d}}_{13}a_1^*\mathrm{v}_2 + \acute{\mathfrak{e}}_{14}a_1^*\mathrm{v}_1^2 + \acute{\mathfrak{e}}_{15}a_1^*\mathrm{v}_2^2 + \mathcal{O}(\mathrm{v}_1, \mathrm{v}_2)^3,\\ \mathrm{v}_2 &= \acute{\mathfrak{d}}_{21}\mathrm{v}_1 + \acute{\mathfrak{d}}_{22}\mathrm{v}_2 + \acute{\mathfrak{d}}_{23}\mathrm{v}_1^2 + \acute{\mathfrak{d}}_{24}\mathrm{v}_1\mathrm{v}_2 + \acute{\mathfrak{e}}_{21}\mathrm{v}_1\mathrm{v}_2 a_1^* + \acute{\mathfrak{e}}_{22}a_1^*\mathrm{v}_1\\ &\quad + \acute{\mathfrak{e}}_{23}a_1^*\mathrm{v}_2 + \acute{\mathfrak{d}}_{24}a_1^*\mathrm{v}_1^2 + \acute{\mathfrak{d}}_{25}a_1^*\mathrm{v}_2^2 + \mathcal{O}(\mathrm{v}_1, \mathrm{v}_2)^3,\end{aligned} \tag{3.9}$$

where

$$\acute{\mathfrak{d}}_{11} = r_1 - \frac{2r_1 \mathrm{u}_1^*}{\xi(1-\eta)} - \frac{\zeta(1-a_1)\mathrm{u}_1^*\mathrm{u}_2^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2},$$

$\acute{\mathfrak{d}}_{12} = -\frac{\zeta(1-a_2)\mathrm{u}_1^*}{\mathrm{u}_1^*+a_2\zeta(1-a_1)\mathrm{u}_2^*}$, $\acute{\mathfrak{d}}_{13} = -a_2 - \frac{\zeta(1-a_1)^2\mathrm{u}_1^*}{(\mathrm{u}_1^*+a_2\zeta(1-a_1)\mathrm{u}_2^*)^2}$,

$$\acute{\mathfrak{d}}_{14} = \frac{\zeta(1-a_1)}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*} - \frac{\zeta(1-a_1)^2\mathrm{u}_1^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} + \frac{a_2\zeta^2(1-a_1)\mathrm{u}_1^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2},$$

$\acute{\mathfrak{d}}_{21} = \frac{\zeta(1-a_1)\mathrm{u}_2^*}{\mathrm{u}_1^*+a_2\zeta(1-a_1)\mathrm{u}_2^*}$, $\acute{\mathfrak{d}}_{22} = -r_2 + \frac{\zeta(1-a_1)\mathrm{u}_2^*}{\mathrm{u}_1^*+a_2\zeta(1-a_1)\mathrm{u}_2^*}$, $\acute{\mathfrak{d}}_{23} = \frac{\zeta(1-a_1)^2\mathrm{u}_2^*}{(\mathrm{u}_1^*+a_2\zeta(1-a_1)\mathrm{u}_2^*)^2}$,

$$\acute{\mathfrak{d}}_{24} = \frac{\zeta(1-a_1)}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*} + \frac{\zeta(1-a_1)^2\mathrm{u}_1^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} - \frac{a_2\zeta^2(1-a_1)\mathrm{u}_1^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2},$$

$$\acute{\mathfrak{e}}_{11} = \frac{a_2\zeta^2(1-a_1){\mathrm{u}_1^*}^2\mathrm{u}_2^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} - \frac{\zeta\mathrm{u}_1^*\mathrm{u}_2^*}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*},$$

$$\acute{\mathfrak{e}}_{12} = \frac{a_2\zeta^2(1-a_1)\mathrm{u}_1^*\mathrm{u}_2^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} + \frac{2a_2\zeta^2(1-a_1)\mathrm{u}_1^*\mathrm{u}_2^*}{(\mathrm{u}_1^* + a_1\zeta(1-a_1)\mathrm{u}_2^*)^2} - \frac{\zeta\mathrm{u}_2^*}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*},$$

$$\acute{\mathfrak{e}}_{13} = \frac{a_2\zeta^2(1-a_1){\mathrm{u}_1^*}^2}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} - \frac{\zeta\mathrm{u}_1^*}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*},$$

$\acute{\mathfrak{e}}_{14} = \frac{2a_2\zeta^2(1-a_1)\mathrm{u}_2^*}{(\mathrm{u}_1^*+a_2\zeta(1-a_1)\mathrm{u}_2^*)^2}$, $\acute{\mathfrak{e}}_{15} = \frac{2a_2\zeta^2(1-a_1)\mathrm{u}_1^*}{(\mathrm{u}_1^*+a_2\zeta(1-a_1)\mathrm{u}_2^*)^2}$,

$$\acute{\mathfrak{e}}_{21} = -\frac{a_2\zeta^2(1-a_1){\mathrm{u}_1^*}^2\mathrm{u}_2^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} + \frac{\zeta\mathrm{u}_1^*\mathrm{u}_2^*}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*},$$

$$\acute{\mathfrak{e}}_{22} = -\frac{3a_1\zeta^2(1-a_1)\mathrm{u}_1^*\mathrm{u}_2^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} + \frac{\zeta\mathrm{u}_1^*}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*},$$

$$\acute{\mathfrak{e}}_{23} = -\frac{a_2\zeta^2(1-a_1){\mathrm{u}_1^*}^2}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} + \frac{\zeta\mathrm{u}_1^*}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*},$$

$\acute{\mathfrak{e}}_{24} = -\frac{2a_2\zeta^2(1-a_1)\mathrm{u}_2^*}{(\mathrm{u}_1^*+a_2\zeta(1-a_1)\mathrm{u}_2^*)^2}$,

$$\acute{\mathfrak{e}}_{25} = -\frac{2a_2\zeta^2(1-a_1)\mathrm{u}_1^*}{(\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*)^2} + \frac{\zeta}{\mathrm{u}_1^* + a_2\zeta(1-a_1)\mathrm{u}_2^*}.$$

The characteristic equation associated with the linearization of system (3.9) at $(\mathrm{v}_1, \mathrm{v}_2) = (0, 0)$ is given by

$$\lambda = \frac{\mathrm{Tr}(J(a_1^*)) + \sqrt{\mathrm{Tr}(J(a_1^*))^2 - 4\mathrm{Det}(J(a_1^*))}}{2}.$$

Correspondingly, when 'a_1' varies in a small neighborhood of $a_1^* = 0$ and there we have

$$|\lambda_{1,2}| = \mathrm{Det}(J(a_1^*))^{\frac{1}{2}},$$

and

$$l = \frac{\mathrm{d}|\lambda_{1,2}|}{\mathrm{d}a_1}\bigg|_{a_1=a_1^*} = -\frac{1}{2}\left\{r_1\left(1 - \frac{2(1+r_2)}{\hbar - a_2\zeta(1+r_2)}\right)\right\}^{-\frac{1}{2}} \neq 0. \tag{3.10}$$

In addition it is required that when $a_1^* = 0$, $\lambda_{1,2}^i \neq 1$, $i = 1, 2, 3, 4$, which is equivalent to

$$\mathrm{Tr}(J(0)) \neq -2, -1, 1, 2. \tag{3.11}$$

Next we study the normal form of (3.9) when $a_1^* = 0$. Let $r_1 = \mathrm{Re}(\lambda)$, $\zeta = \mathrm{Im}(\lambda)$, and

$$\Upsilon = \begin{pmatrix} 0 & 1 \\ \zeta & r_1 \end{pmatrix},$$

and use the translation

$$\begin{pmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{pmatrix} = \Upsilon \begin{pmatrix} \tilde{\mathrm{u}}_1 \\ \tilde{\mathrm{u}}_2 \end{pmatrix},$$

the system (3.9) becomes

$$\begin{pmatrix} \tilde{\mathrm{u}}_1 \\ \tilde{\mathrm{u}}_2 \end{pmatrix} \to \begin{pmatrix} r_1 & -\zeta \\ \zeta & r_1 \end{pmatrix}\begin{pmatrix} \tilde{\mathrm{u}}_1 \\ \tilde{\mathrm{u}}_2 \end{pmatrix} + \begin{pmatrix} \tilde{f}(\tilde{\mathrm{u}}_1, \tilde{\mathrm{u}}_2) \\ \tilde{g}(\tilde{\mathrm{u}}_1, \tilde{\mathrm{u}}_2) \end{pmatrix}, \tag{3.12}$$

where,

$$\begin{aligned}\tilde{\varrho}(\tilde{\mathrm{u}}_1, \tilde{\mathrm{u}}_2) = \frac{1}{\mathrm{Det}(\Upsilon)}\Big[&\{(r_1 - \acute{\mathfrak{d}}_{11})\acute{\mathfrak{d}}_{13} - \acute{\mathfrak{d}}_{12}\acute{\mathfrak{d}}_{23}\}\,\tilde{\mathrm{u}}_1^2 \\ &+ \{(r_1 + \acute{\mathfrak{d}}_{11})\acute{\mathfrak{d}}_{14} - \acute{\mathfrak{d}}_{12}\acute{\mathfrak{d}}_{24}\}\tilde{\mathrm{u}}_1\tilde{\mathrm{u}}_2 + \mathcal{O}(|\mathrm{v}_1|, |\mathrm{v}_2|)^4\Big],\end{aligned}$$

$$\begin{aligned}\tilde{\varrho}_2(\tilde{\mathrm{u}}_1, \tilde{\mathrm{u}}_2) = \frac{1}{\mathrm{Det}(\Upsilon)}\Big[&\{(r_1 + \acute{\mathfrak{d}}_{11})\acute{\mathfrak{d}}_{13} - \acute{\mathfrak{d}}_{12}\acute{\mathfrak{d}}_{23}\}\,\tilde{\mathrm{u}}_1^2 \\ &+ \{(r_1 + \acute{\mathfrak{d}}_{11})\acute{\mathfrak{d}}_{14} - \acute{\mathfrak{d}}_{12}\acute{\mathfrak{d}}_{24}\}\tilde{\mathrm{u}}_1\tilde{\mathrm{u}}_2 + \mathcal{O}(|\mathrm{v}_1|, |\mathrm{v}_2|)^4\Big].\end{aligned}$$

Now, we can see that (3.12) is exactly in the form on the center manifold, in which the coefficient l_1 [31, 32] is given by

$$l_1 = -\mathrm{Re}\left[\frac{(1-2\bar{\lambda})\bar{\lambda}^2}{1-\lambda}\iota_{11}\iota_{20}\right] - \frac{1}{2}\left(\|\iota_{11}\|^2 + \|\iota_{02}\|^2\right) + \mathrm{Re}(\bar{\lambda}\iota_{21}), \tag{3.13}$$

where,

$$\iota_{20} = \frac{1}{8}\Big[(\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_1} - (\acute{\varrho}_1)_{\tilde{u}_2\tilde{u}_2} + 2(\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_2} + i\big((\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_1} - (\acute{\varrho}_2)_{\tilde{u}_2\tilde{u}_2} - 2(\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_2}\big)\Big],$$

$$\iota_{11} = \frac{1}{4}\Big[(\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_1} + (\acute{\varrho}_1)_{\tilde{u}_2\tilde{u}_2} + i\big((\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_1} + (\acute{\varrho}_2)_{\tilde{u}_1\tilde{y}}\big)\Big],$$

$$\iota_{02} = \frac{1}{8}\Big[(\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_1} - (\acute{\varrho}_1)_{\tilde{u}_2\tilde{u}_2} + 2(\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_2} + i\big((\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_1} - (\acute{\varrho}_2)_{\tilde{y}\tilde{y}} - 2(\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_2}\big)\Big],$$

$$\iota_{21} = \frac{1}{16}\Big[(\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_1\tilde{u}_1} + (\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_2\tilde{u}_2} + (\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_1\tilde{u}_2} + (\acute{\varrho}_2)_{\tilde{u}_2\tilde{u}_2\tilde{u}_2} + i\big((\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_1\tilde{u}_1} + (\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_2\tilde{u}_2} - (\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_1\tilde{u}_2} - (\acute{\varrho}_1)_{\tilde{u}_2\tilde{u}_2\tilde{u}_2}\big)\Big],$$

and

$$(\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_1} = \frac{2\{(r_1 - \acute{\mathfrak{d}}_{11})\acute{\mathfrak{d}}_{13} - \acute{\mathfrak{d}}_{12}\acute{\mathfrak{d}}_{23}\}}{\mathrm{Det}(\Upsilon)},$$

$$(\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_2} = \frac{(r_1 - \acute{\mathfrak{d}}_{11})\acute{\mathfrak{d}}_{14} - \acute{\mathfrak{d}}_{12}\acute{\mathfrak{d}}_{24}}{\mathrm{Det}(\Upsilon)},$$

$$(\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_1} = \frac{2\{(r_1 + \acute{\mathfrak{d}}_{11})\acute{\mathfrak{d}}_{13} - \acute{\mathfrak{d}}_{12}\acute{\mathfrak{d}}_{23}\}}{\mathrm{Det}(\Upsilon)},$$

$$(\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_2} = \frac{(r_1 + \acute{\mathfrak{d}}_{11})\acute{\mathfrak{d}}_{14} - \acute{\mathfrak{d}}_{12}\acute{\mathfrak{d}}_{24}}{\mathrm{Det}(\Upsilon)},$$

$(\acute{\varrho}_1)_{\tilde{u}_2\tilde{u}_2} = (\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_1\tilde{u}_2} = (\acute{\varrho}_1)_{\tilde{u}_2\tilde{y}\tilde{u}_2} = (\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_1\tilde{u}_2} = (\acute{\varrho}_1)_{\tilde{u}_1\tilde{u}_2\tilde{u}_2} = (\acute{\varrho}_2)_{\tilde{u}_2\tilde{u}_2} = (\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_1\tilde{u}_1} = (\acute{\varrho}_2)_{\tilde{u}_2\tilde{u}_2\tilde{u}_2} = (\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_1\tilde{u}_2} = (\acute{\varrho}_2)_{\tilde{u}_1\tilde{u}_2\tilde{u}_2} = 0.$

Theorem 3.2. *If condition* (3.10) *and* (3.11) *hold and* $l_1 \neq 0$*, then system* (3.6) *undergoes a Hopf-bifurcation at the equilibrium point* $\mathfrak{E}^*(u_1^*, u_2^*)$ *when the parameter '*a_1*' varies in the small neighborhood of the origin. Moreover, if* $l_1 < 0\,(> 0)$*, then an attracting (repelling) invariant closed curve bifurcates from the fixed point* $\mathfrak{E}^*(u_1^*, u_2^*)$*.*

4 Numerical Simulations

In this section, we ar going to present some numerical results for some particular values of the the parameters associated with the model system (1.2).

Example 4.1. *Base on the system* (1.2)*, We consider the*

$$\begin{aligned}(u_1)_{n+1} &= (u_1)_n\left[r_1\left(1 - \frac{(u_1)_n}{0.2(1-\eta)}\right) - \frac{0.7(1-a_1)(u_2)_n}{(u_1)_n + 0.7a_2(1-a_1)(u_2)_{n+1}}\right],\\ (u_2)_{n+1} &= (u_2)_n\left[\frac{2.8(1-a_1)(u_1)_n}{(u_1)_n + 0.7a_2(1-a_1)(u_2)_n} - r_2\right],\end{aligned} \tag{4.1}$$

with $\xi = 0.5$, $\zeta = 0.3$, $\hbar = 2$, $\sigma_{HV} = 1$ *and* $\eta = 0.1$, $r_1 = 2.5$, $r_2(\mathrm{x}) = 3 + 0.2\sin\mathrm{x}$, $a_1(\mathrm{x}) = 2.5{+}0.4\sin\mathrm{x}$, $a_2(\mathrm{x}) = 5{-}0.4\sin\mathrm{x}$, $\eta(\mathrm{x}) = 2{+}0.3\sin\mathrm{x}$. *By applying the Equation* (2.2), *we obtain* u_1^*, u_2^* *for all* $\mathrm{x} \in [\tau_1, \tau_2] = [0, 28]$ *and* $\Delta_1 = 1 - \frac{r_2}{\hbar\zeta(1-a_2r_1)}$, $\Delta_2 = \hbar\zeta(1 - a_1)$. *One can check the numerical results in Table 1 and can see 2D plot of them in Fig. 1.*

Table 1: Numerical values of u_1^*, u_2^* in Example 4.1 $\forall\mathrm{x} \in [0, 28]$.

x	u_1^*	u_2^*	$a_1(\mathrm{x})$	Δ_1	$r_2(\mathrm{x})$	Δ_2
0.00	−0.1964	0.0056	0.2000	1.1905	2.0000	2.2400
0.02	−0.1955	0.0054	0.2070	1.1902	1.9895	2.2205
0.03	−0.1946	0.0052	0.2140	1.1898	1.9791	2.2009
0.05	−0.1936	0.0050	0.2209	1.1895	1.9686	2.1814
0.07	−0.1927	0.0048	0.2279	1.1892	1.9581	2.1619
0.09	−0.1918	0.0046	0.2349	1.1889	1.9477	2.1424
0.10	−0.1909	0.0044	0.2418	1.1885	1.9373	2.1229
0.12	−0.1900	0.0042	0.2487	1.1882	1.9269	2.1035
0.14	−0.1890	0.0040	0.2557	1.1879	1.9165	2.0841
0.16	−0.1881	0.0038	0.2626	1.1875	1.9061	2.0648
0.17	−0.1872	0.0036	0.2695	1.1872	1.8958	2.0455
0.19	−0.1863	0.0034	0.2763	1.1869	1.8855	2.0263
0.21	−0.1854	0.0032	0.2832	1.1865	1.8753	2.0071
0.23	−0.1845	0.0030	0.2900	1.1862	1.8650	1.9881
0.24	−0.1836	0.0028	0.2968	1.1858	1.8548	1.9690
0.26	−0.1828	0.0026	0.3035	1.1854	1.8447	1.9501
0.28	−0.1819	0.0024	0.3103	1.1851	1.8346	1.9313
0.30	−0.1810	0.0022	0.3169	1.1847	1.8246	1.9125
0.35	−0.1784	0.0016	0.3368	1.1836	1.7948	1.8569
0.37	−0.1776	0.0014	0.3433	1.1833	1.7850	1.8386
0.38	−0.1768	0.0011	0.3498	1.1829	1.7752	1.8204
0.40	−0.1759	0.0009	0.3563	1.1825	1.7656	1.8024
0.42	−0.1751	0.0007	0.3627	1.1821	1.7560	1.7845
0.44	−0.1743	0.0005	0.3690	1.1818	1.7464	1.7667
0.45	−0.1735	0.0003	0.3753	1.1814	1.7370	1.7490
0.47	−0.1727	0.0001	0.3816	1.1810	1.7276	1.7315

5 Conclusion

In this paper we investigate the behavior of the predator prey system as a discrete time scale where functional response dependents on predator density in different way. It is seen that if $0 < a_1 < 1 - \frac{r_2}{\hbar\zeta(1-a_2r_1)}$ and $0 < r_2 < \hbar\zeta(1 - a_1)$, then the system (2.1) will have feasible coexistence equilibrium point. In section 2.2, analytically we have obtained several condition for stability of the equilibrium points. Also we have studied the conditions for which an equilibrium point will be called as source, sink and saddle. In section 3, sufficient conditions for flip bifurcation as well as hopf-bifurcation has been studied respectively. These results reveal far richer dynamics of the discrete model compared to the continuous model. This complex phenomena may be essential for the intraspecific competitive between the predator and the prey.

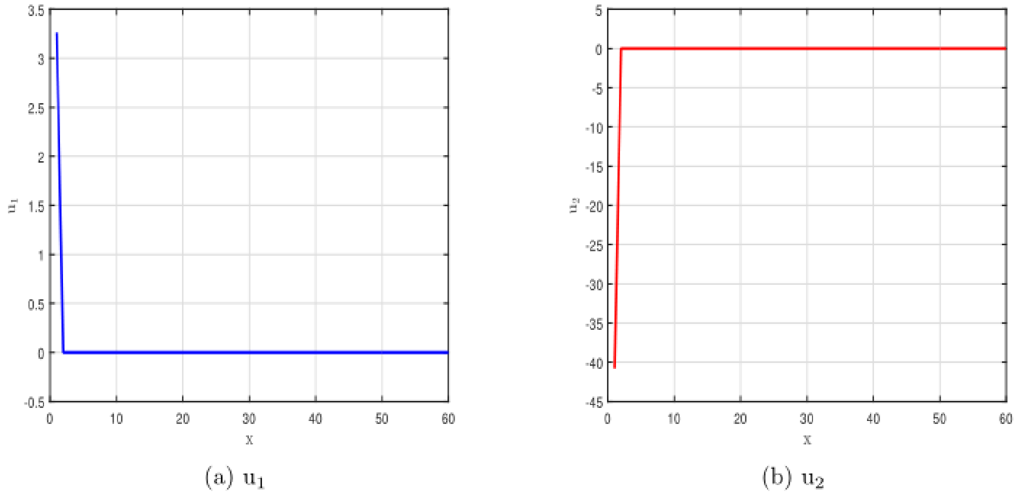

Figure 1 Graphical representation of u1, u2 in Example 4.1.

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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