

Original Research Article

Graphs with small third largest Laplacian eigenvalue

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Abstract: Let $L(G)$ be the Laplacian matrix of G . In this paper, all connected graphs with third largest Laplacian eigenvalue no more than ℓ are characterized, where $\ell = 1.3820$ is the smallest root of the equation $x^2 - 5x + 5 = 0$ is characterized. Moreover, these results are used to characterize all connected graphs with the third largest Laplacian eigenvalue is 1.

Keywords: Laplacian matrix; characteristic polynomial; Laplacian eigenvalue

1. Introduction

In this paper, all graphs are undirected connected graphs without loops and multiple edges. Let $G = (V, E)$ denote a simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. We denote the diagonal matrix of vertex degrees by $D = D(G)$ and denote the adjacency matrix by $A(G)$. Then $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G (for example, see). Obviously, $L(G)$ is a positive semi-definite symmetric matrix. We use $(G; \mu)$ to denote the Laplacian characteristic polynomial of $L(G)$. Its eigenvalues will be called the *Laplacian eigenvalues* of graph G . They will be denoted and ordered by $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G) = 0$. If $\{v_6, v_7, \dots, v_n\} \in V(G)$, let $L_v(G)$ be the principal submatrix of $L(G)$ formed by deleting the row and column corresponding the vertex v . Let U be a set of vertices in $V(G)$. Denote by $G[U]$ the induced subgraph by U . Let $G - e$ be the graph which is obtained from G by removing edge e from the edge set of G . Let $d(v)$ denote the degree of the vertex $v \in V(G)$.

In this paper, all connected graphs with third largest Laplacian eigenvalue no more than ℓ are characterized, where $\ell = 1.3820$ is the smallest root of the equation $x^2 - 5x + 5 = 0$ is characterized. Moreover, these results are used to characterize all connected graphs with the third largest Laplacian eigenvalue is 1.

2. Preliminary

We first present some well known results, which will be used later.

Lemma 2.1 Let G be a simple graph with order n , and let H be a subgraph of G (not necessarily an induced subgraph) with order m ($m \leq n$). Then for $i = 1, 2, \dots, m$, we have $\mu_i(G) \geq \mu_i(H)$.

Lemma 2.2 Let A be a Hermitian matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ and B be one of its principal submatrix with eigenvalues $\mu'_1 \geq \mu'_2 \geq \dots \geq \mu'_n$. Then the inequalities $\mu_{n-m+i} \leq \mu'_i \leq \mu_i$ ($i = 1, 2, \dots, m$) hold.

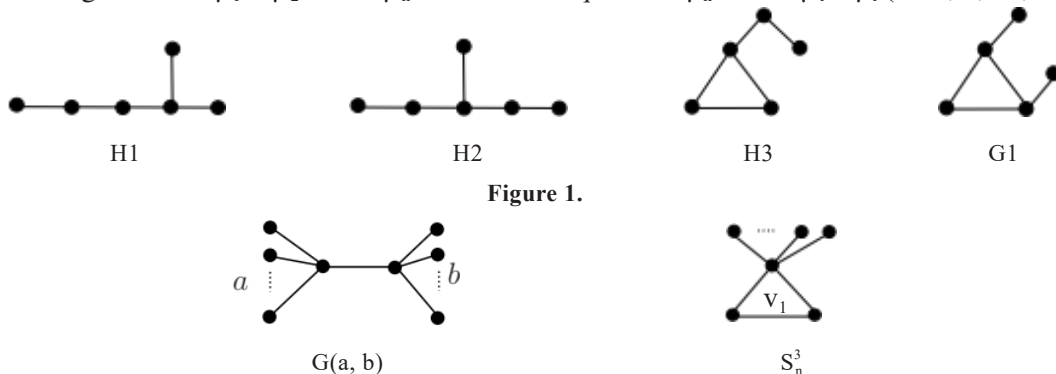


Figure 1.

Figure 2.

We use $L_v(G)$ to denote the principal submatrix of $L(G)$ which is obtained by deleting the row and the

column of $L(G)$ corresponding to the vertex v .

Lemma 2.3 Let S_n be a star of order n ($n \geq 3$). Then $\mu_3(S_n) \leq 1$.

Proof. By a direct calculation, we have $\det(\mu I - L(S_n)) = \mu(\mu - n)(\mu - 1)^{n-2}$.

Then $\mu_3(S_n) \leq 1$.

Lemma 2.4 Let $G(a, b)$ be a graph of order $n = a + b + 2$ ($a \geq 0, b \geq 0$). Then $\mu_3(G(a, b)) \leq 1$.

Proof. By a direct calculation, we have $\det(\mu I - L_{V_1 V_2}(G(a, b))) = (\mu - 1)^{n-2}$.

By lemma 2.2,

$\mu_3(G(a, b)) \leq \mu_1(L_{V_1 V_2}(G(a, b))) = 1$.

Then $\mu_3(G(a, b)) \leq 1$.

Lemma 2.5 Let S_n^3 be a graph of order n ($n \geq 3$). Then $\mu_3(S_n^3) \leq 1$.

Proof. By a direct calculation,

$\det(\mu I - L_{V_1}(S_n^3)) = (\mu - 1)^{n-2}(\mu - 3)$.

By lemma 2.2, we have $\mu_3(S_n^3) \leq \mu_2(L_{V_1}(S_n^3)) = 1$.

3. Main results

Theorem 3.1 Let G be a tree with order n . If $\mu_3(G) \leq \ell$, then G must be a subgraph of the graphs in $\{S_n, G(a, b), P_5\}$.

Proof. By lemma 2.1, P_6 is a forbidden subgraph of G . Then the diameter of G is less than 5. Clearly, $2 \leq \text{diam}(G) \leq 5$. If $\text{diam}(G) = 2$, then it is easy to see that $G \cong S_n$, as desired.

If $\text{diam}(G) = 3$, then G contains a path P_4 as its subgraph, say $P_4 = v_1 v_2 v_3 v_4$. Let $U = \{v_5, v_6, \dots, v_n\} \subseteq V(G)$. Obviously, any vertex in U is adjacent to neither v_1 nor v_4 . Then there exists one edge between any connected component of $G[U]$ and v_i ($i = 1, 2$). Since $\text{diam}(G) = 3$, it follows that $G[U]$ is a graph with $n-4$ isolated vertices. Therefore, $G \cong G(a, b)$.

Suppose $\text{diam}(G) = 4$. In this case, G must contain the path P_5 as its subgraph. Let $U = \{v_6, v_7, \dots, v_n\} \subseteq V(G)$. Since P_6, H_1, H_2 are forbidden subgraphs of G , there is no edge between $G[U]$ and v_i ($i = 1, 2, \dots, 5$). Then $G \cong P_5$.

Theorem 3.2 Let G be a connected graph with n ($n \geq 3$) vertices and m ($m \geq n$) edges,

if $\mu_3(G) \leq 1$. Then G must be a subgraph of the graphs in $\{S_n^3, G_1, C_5\}$.

Proof. Since P_6 is a forbidden subgraph of G , it follows that $c(G) \leq 5$. By lemma 2.1, C_4 is the forbidden subgraph of G , we have that $c(G) \neq 4$. Clearly, $3 \leq c(G) \leq 5$.

Suppose $c(G) = 3$. Let $C_3 = v_1 v_2 v_3 v_1$ be a cycle of order 3 in G . Let $U = \{v_4, v_5, \dots, v_n\} \subseteq V(G)$. Since H_2 is a forbidden subgraph of G , there is at least one vertex of $\{v_1, v_2, v_3\}$ whose degree is two. If there exists only one vertex v in $\{v_1, v_2, v_3\}$ such that $d_G(v) > 2$. Without loss of generality, let $d(v_1) > 2$ and $d(v_2) = d(v_3) = 2$. Then there exists one edge between any component of $G[U]$ and v_1 . Since H_3 is a forbidden subgraph of G , it follows that $G[U]$ is a graph with $n-3$ isolated vertices. Therefore,

Then $G \cong S_n^3$. If there exist two vertex in $\{v_1, v_2, v_3\}$ whose degrees are more than two.

Without loss of generality, let $d(v_1) > 2, d(v_2) > 2$. Since H_1, H_3 are forbidden subgraphs of G , it follows that $G \cong G_1$.

Suppose $c(G) = 5$. Let $C_5 = v_1 v_2 \dots v_5 v_1$ be a cycle of order 5 in G . Let $U = \{v_6, v_7, \dots, v_n\} \subseteq V(G)$. Since H_1 is forbidden subgraph of G , there is no edge between $G[U]$ and v_i ($i = 1, 2, \dots, 5$). So, the order of G is 5. Since H_4 is a forbidden subgraph of G , it follows that $G \cong C_5$.

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