Limit and infinitesimal analysis

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Abstract: Limit is a very important basic knowledge in calculus, the definition of limit is a simple logical proposition, which involves the existence, arbitrariness, and inequality problems. In this paper, one gives some explanations of the limit, the relationship between the limit and the infinitesimal are demonstrated, from the infinitesimal order to understand the limit.

Key words: limit; Infinitesimal; Order; Taylor's theorem

1. Introduction

The limit is an extremely important concept in calculus by which the continuity and derivative of a function can be defined. Infinitesimal and infinitesimal orders help you understand the limits of functions and how to compare "orders of magnitude" between limits.

Scientific numeration is very convenient when people need to deal with very large numbers. The idea of counting is to count and compare the size of the book by defining the bits of the number, without writing down many zeros or counting how many zeros there are. If you add or multiply the two numbers, it's not very difficult.

Logarithms is invented and used in a similar way to solve the problem of exponents. The estimation of infinitely small quantities, infinitely large quantities, and orders is the scientific notation of calculus. The words infinite approach and approximation seem very loose, but they are a kind of visualization of the strict expression, and the original strict expression is not a problem.

1. let's look at the definition of the limit: $\lim_{x \to a} f(x) = A$.

The definition of the limit is: for any $\varepsilon > 0$ there exists a $\delta > 0$ such that when $|x-a| < \delta$ is true ,the inequality $|f(x)-A| < \varepsilon$ is true at that time.

The definition of the limit is a naive logical statement, the judgment of the inequality. For any $\varepsilon > 0$,there exists one $\delta > 0$ such that the inequality $|f(x)-A| < \varepsilon$, $\varepsilon > 0$ is true when the inequality of about $\delta > 0$, $|x-a| < \delta$ is true.

If such a condition is satisfied, then the function has a limit. No matter how complicated the problem is, what one really cares about is whether the inequality holds or not. As you can imagine, this is a simple algorithmic function, and if you pass any positive value $\varepsilon > 0$ to it, it always returns $\delta > 0$.

If one look at the limit from the point of view of an infinitesimal and with transformation $g(x) = f(x) - A$, one can get an equivalent definition of the limit that is for any $\varepsilon > 0$, exists $\delta > 0$ such that $|x-a| < \delta$, at that time, the inequality $|g(x)| < \varepsilon$ holds. Currently, one call $g(x)$ infinitesimal.

Because of this, all studies of limits become equivalent studies of infinitesimals, so limit theory can also be called infinitesimal analysis. Infinitesimal itself is a variable, a function, not a constant, even if the only constant infinitesimal one will treat it as a constant function. Therefore, one should use dynamic thinking to look at and understand infinitesimal, rather than treating it as a small number.

2. Infinitesimal, infinitesimal order

2.1 Equivalent infinitesimals

Consider an important limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ that plays an important role in calculus, and at this point one can express the limit as $\sin x \sim x$, this is equivalent to $\frac{\sin x}{x} \to 1$. Here one can say that at that time, the function $\sin x$ and x are very close when $x \to 0$, and the proximity is equivalent, that is, the quotient between the two functions is close to 1. This argument may involve some computational skills, but the logic problem is still expressed in a very simple way.

Strictly speaking, if for any two functions $f(x), g(x)$, if there is $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$, it can be written as $f(x) \sim g(x)$ where $-\infty < a < +\infty$. If $f(x) \to 0$ or $f(x) \to \infty$, it is computing an infinitely small quantity or an infinitely large quantity, then it's called $f(x)$ is an equivalent infinitesimal (large) quantity of $g(x)$. Usually only infinitesimal is considered, for infinity, it can be understood as the reciprocal of non-zero infinitesimal. For relation ~, the following three properties are satisfied:

1. Reflexivity: $f \sim f$
2. Symmetry: $f \sim g \Leftrightarrow g \sim f$
3. Transitivity: if $f \sim g, g \sim h \Rightarrow f \sim h$

If these three properties are satisfied, it is called an equivalence relation. For example, the equality relation of real numbers is an equivalence relation. If one only consider the equivalence relation, the ratio 1 is too strict, so one do some generalization of the equivalence relation, giving the concept of the same order infinitesimal: if $\lim_{x\to a}\frac{f(x)}{g(x)}=A$, then $f(x)$ and $g(x)$ are said to be the same order infinitesimal. In fact, dividing A both sides in the above definition can be converted to the definition of an equivalent infinitesimal, i.e. $f(x)\sim Ag(x)$

2.2 Application of infinitesimals

Why is the concept of infinitesimal introduced? While the actual function computation may be tedious (involving trigonometric functions, inverse trigonometric functions, etc.), there is one kind of function computation that has always been very trivial: polynomials functions. This kind of function can even calculate the value directly. If one can reduce a complex function to a power function, things are much simpler, and the error introduced by equivalent infinitesimals is acceptable. Or let's try to express the "order of magnitude" of a function as a poorer function, and this "quantity" refers to the approximation speed at some point. It doesn't have to be a power function, of course, but it can be done the other way around, to a different but better calculated limit. A classic example is explained below.

Example 1: Finding the limit $\lim_{x\to 0}\frac{\ln(1+x^3)}{x}$

Solution: According to the principle of equivalent infinitesimals, one have. $\ln(1+x^3)\sim x^3$ .Thus have $\lim_{x\to 0}\frac{\ln(1+x^3)}{x}=\lim_{x\to 0}\frac{x^3}{x}=0$.

In this problem one rely on a very simple conclusion: two functions, that is, they are of the same order of magnitude, and then one assign the old limit to the new ratio. $\ln(1+x^3)\sim x^3$ at $x=0$ .Since one are trying to "reduce" the limit of a continuous function to a power function, one can define the order of the limit in terms of the order of the power function. That is, if a constant α is known and exists $\lim_{x\to a} f(x)=0$ such that $\lim_{x\to a}\frac{f(x)}{x^\alpha}=A\neq 0$, i.e., $f(x)\sim A(x-a)^\alpha$ there is said $f(x)$ to be an infinitesimal of order at $x=a$ nearby. This α is like an order of magnitude in scientific decimal notation, which doesn't give us the exact information, but it gives us enough information in some circumstances, which is exactly what one want. Order is a great tool, but not all functions have orders. Some functions don't have orders. For example, functions $f(x)=x\sin\frac{1}{x}, x=0$ have no order in their vicinity. This is because the limit of theta does not exist. $\lim_{x\to 0}\frac{x\sin\frac{1}{x}}{x^\alpha}$ The reader can discuss with the size of α and 1, verifying that the limit does not exist.

Misestimating an infinitesimal class can cause a lot of trouble, such as the two functions x^2, x^3 are still very different. If you estimate the wrong order when calculating the limit, you may get the wrong result. The equivalence relation does not tell us anything about addition and subtraction. For example, the equivalence relation only defines the relationship between two functions, multiplication or division, but it is missing for addition and subtraction. If you don't pay attention to this, the limit $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$ will be miscalculated. If one use the equivalent infinitesimal directly, one get

$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{x-x}{x^3}=0$ that this result is wrong. The reason this is wrong is precisely because the principle of equivalent substitution is not followed in addition and subtraction. The result of the limit is 1/3, that is, the order of the numerator is 3, which cannot be obtained from the equivalent substitution.

2.3 Notation o, O

For infinitesimal addition and subtraction cannot correctly give the exact information of the order, in the comparison of different orders of infinitesimal problems, give the comparison definition of infinitesimal order:

Definition 2.3 If satisfied $\lim_{x\to a}\frac{f(x)}{g(x)}=0$, take note $f(x)=o(g(x))(x\to a)$.

If $f(x)\to 0$, then one say $f(x)$ is higher order infinitesimal smaller than $g(x)$.

For example, $\alpha>1$, one can write. $x^\alpha=o(x)$. In the definition of higher order infinitesimal, one use the function $g(x)$ as a reference, then the convergence rate of the function $f(x)$ is faster than $g(x)$. The higher order infinitesimal is transitive, which brings us convenience in judging the size of the infinitesimal order.

If there is $f(x)=o(1),(x\to a)$,then there is $f(x)\to 0$. Of course, it's awkward to use the notation o here, but there's really no problem with using 1 as the denominator! At this time, one look at the convergence rate compared with the constant function, then the function convergence is convergence, not convergence is not convergence. Let's look at the meaning of another symbol O .

Definition 2.4 $f(x)=O(g(x))(x\to a)$ is defined that there exists some a de-centered neighborhood and a non-negative constant M such that $\left|\frac{f(x)}{g(x)}\right|\leq M$ is established over this neighborhood.

The "de-centered neighborhood" here is not a complicated concept. When you're dealing with limits, $f(x), g(x)$ might not be a limit at a , so you're talking about a interval, and you're picking a out, which is $(a-\delta,a)\cup(a,a+\delta)$. This is the roughest estimate, but it's also very convenient. It may be a bit cumbersome to calculate the limit of the ratio but knowing that the ratio is bounded saves us a lot of trouble.

In Fourier series, one study a class of "moderately decreasing" functions, and one need the function $f(x)$ to satisfy it. $f(x)=O(\frac{1}{1+|x|^\alpha})$ where $\alpha>1$.

If you look at the previous function, which has no order, you can see $x\sin\frac{1}{x}=O(x)$.

So, O can give a qualitative result of the order of a function in some cases, which is useful in the study of infinitesimals with no order.

o, O , ~ is a “reference” action symbol that abstracts the conclusions of the ratios and limits satisfied by the specified function, and uses these conclusions to represent a function, thereby indirectly calculating the original limit. This is like simplifying the problem down to a few decimal places.

3 Taylor’s Theorem

Taylor’s formula can be understood as describing values near a point in terms of the value of a function at that point. For sufficiently smooth functions, the structure of the function near them can be constructed from the values of the derivatives of each order of the function at a certain point. Taylor’s formula is to use these derivative values as coefficients to construct a polynomial to approximate the value of the function in the neighborhood at this point. This polynomial is called the Taylor polynomial. Taylor’s theorem gives the remainder -- the difference between the polynomial and the actual function value.

Taylor’s theorem

Theorem 3.1 (Taylor’s Formula) If a function $f(x)$ at a point x_0 has an $n-th$ order derivative, then

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+...+\frac{f^n(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n)$$

This theorem gives a property of order of the function $f(x)$ the nearby x_0. In contrast to the equivalent infinitesimals, $f(x)$ can be vieoned as an “approximation to the n decimal place”. This makes it easier to deal with the limit problem, so you don’t have to worry about adding and subtracting infinitesimals anymore, because you use theorem 3.1 to approximate the numerator by estimating the order of the denominator in advance. Let’s look at the following example.

Example: $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\frac{x+\frac{1}{3}x^3-(x-\frac{1}{3!}x^3)}{x^3}=\frac{1}{2}$. In this problem, one can see that the denominator is an infinitesimal of order 3, so according to the order of the denominator, if the order of the numerator is preserved to order 3, because the higher order infinitesimal is a higher order infinitesimal for the denominator and can be ignored in the limit calculation. This is an important meaning of theorem 3.1. To further investigate the properties of functions, Taylor’s theorem is given as follows in a more quantitative way.

Theorem 3.2 (Taylor’s theorem) If a function $f(x)$ has continuous derivatives up to n orders at a point x_0 , then there is $n+1$ order derivative at internal (a,b) , then there exist at least $\xi\in(a,b)$ such that

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+...+\frac{f^n(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

As can be seen from the conditions of Taylor’s theorem, the conditions of Taylor’s theorem are much higher than that of Taylor’s formula, so the conclusion obtained is stronger. Using Taylor’s theorem, the function $f(x)$ can be approximated quantitatively by polynomial function. In the theory of infinite series, Taylor’s theorem can be studied more deeply, and a special class of Taylor series, also known as power series, can be obtained. Taylor series have no remaining terms.

4 Conclusions

This paper gives the relationship between limit functions and infinitesimals and approximates the properties of functions near a certain point by introducing the concepts of equivalence and same order of infinitesimals. The disadvantage of infinitesimals is that they cannot be added or subtracted. Based on this, higher order approximations of functions are given Taylor’s formula and Taylor’s theorem. Taylor’s formula approximates a function of higher order, which can be used when it is not necessary to know the remainder of the function at that point. If a higher order remainder is needed to estimate and calculate, then Taylor’s theorem can be used.

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Project Information: Higher Mathematics Teaching Team, No. 2022JXTD021; The Theory of Limits and the Study of Infinite Small Quantities, J20230220