Original Research Article

Existence of ground state solutions of critical Schrödinger equation with singular potential

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Abstract: In the present paper, we are interested in the following Schrödinger equation with singular potential v is a singular potential with parameter 0 < α < 1. Under some reasonable assumptions on v and λ, we establish the existence of ground state solutions. Compared with the recent results obtained in [8], we extend the scope (1,2) of α to (0,1).

Keywords: Schrödinger equation; Singular potential; Ground state solutions

1. Introduction

This paper is concerned with the following class of elliptic problems

$$-\Delta u + V(|x|)u = |u|^{2^*-2}u + |u|^{q-2}u + \lambda|u|^{2^*_\alpha-2}u,\ x \in \mathbb{R}^N, \tag{1.1}$$

where $N = 3, 2^* := \frac{2N}{N-2}, 2^*_\alpha := 2 + \frac{4\alpha}{2N-2-\alpha}, 4 < q < 2^*, \lambda > 0$ is a positive parameter, V is a singular potential with parameter and satisfies the following assumptions:

with parameter $0 < \alpha < 1$ and satisfies the following assumptions:

(V_1) there exists $A > 0$ such that $V(s) \geq \frac{A}{s^\alpha}$ for almost every $s > 0$;

$(V_2) V \in L^1(a, b)$ for some (a, b) with $b > a > 0$;

(V_3) There exists $B \in (A, \infty)$ such that $V(s) \leq \frac{B}{s^\alpha}$ for almost every $s > 0$.

Obviously, the typical functions satisfying $(V_1) - (V_3)$ is of the form $\frac{A}{|x|^\alpha}$, where $\alpha, A > 0$ are real constants. In this sense, taking $\alpha = 2$ and the nonlinearity being the pure-critical case $f(u) = |u|^{2^*-2}u$, the author [11] showed that the behavior of solutions for (1.1) with $N \geq 3$ heavily depends on the changes of the parameter A. Subsequently, the authors [4] investigated the case that $V(|x|) = \frac{A}{|x|^\alpha}$ with $A \in \mathbb{R}, \alpha > 0$ and $f(u) = |u|^{q-2}u$ with $q > 2$. Combining the moving planes and the moving spheres methods, they demonstrated that how the existence of positive solutions of (1.1) can be influenced by the choice of the parameters A, α and q. Since then, problems like (1.1) have received continuous attention, see [1, 2, 3, 4, 6, 7, 8, 9, 10] and the references listed therein. Actually, problems like (1.1) can model the stationary states of reaction diffusion equations in population dynamics ([5]) and also arise in many branches of mathematical physics, such as nonlinear optics, plasma physics, condensed matter physics and cosmology. The motivation of our present paper is the following result obtained in [8], which is also given in 7.

Theorem A.([8, Theorem 1.1]) Assume that $q \in \left(\frac{2N-4+2\alpha}{N-2}, 2^*\right)$ and conditions $(V_1)-(V_3)$ hold. Let

$$\begin{cases} N=3, & \alpha \in (1,2); \\ N \geq 4, & \alpha \in (0,2), \end{cases}$$

and

$$\lambda^* := \left(\frac{\left(\frac{1}{2}-\frac{1}{2_\alpha^*}\right) S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-2}}}{\left(\frac{1}{2}-\frac{1}{2^*}\right) S_\alpha^{\frac{2^*}{2^*-2}}} \right) \geq \lambda > 0, \tag{1.2}$$

where S and S_α are defined in (2.1) and (2.2). Then equation (1.1) has a ground state solution $u \in W_{\mathrm{rad}}^{1,2}(\mathbb{R}^N, V)$. Moreover, $u \in L^\infty(\mathbb{R}^N)$ and $u \in C^1(\mathbb{R}^N \setminus \{0\})$.

The above Theorem A states that, when $N=3, \alpha$ is limited to the interval $(1,2)$. Therefore, an interesting issue is to relax this restrict and prove that (1.1) also admits ground state solutions if $N=3$ and $\alpha \in (0,1]$ just like the case $N \geq 4$. Motivated by this observation, we still focus our attention on problem (1.1) and complement the above Theorem A in the following sense.

Theorem 1.1. *Assume that conditions $(V_1)-(V_3)$ hold. Let $N=3, \alpha \in (0,1), q \in (4, 2^*)$ and $0<\lambda<\lambda^*$, (1.1) has a ground state solution $u \in W_{rad}^{1,2}(\mathbb{R}^N, V)$. Moreover, $u \in L^\infty(\mathbb{R}^N)$ and $u \in C^1(\mathbb{R}^N \setminus \{0\})$.*

Remark 1.2. When $N=3$ and $\alpha \in (0,1)$, it is obvious to see that $\frac{2N-4+2\alpha}{N-2} = 2+2\alpha < 4$. In other words, our conclusion could not cover the range between $2+2\alpha$ and 4 , that is, $(2+2\alpha, 4]$. Meanwhile, from Theorem A and our Theorem 1.1, we see that the case $\alpha = 1$ is also not involved.

To achieve Theorem A and our Theorem 1.1, the next embedding plays an essential role.

Proposition B. (8 , Proposition 2]) Suppose that $N \geq 3$ and condition $(V_1)-(V_3)$ hold. Then the following continuous embeddings hold:

$$\begin{cases} W_{rad}^{1,2}(\mathbb{R}^N, V) \hookrightarrow L^r(\mathbb{R}^N), & r \in [2_\alpha^*, 2^*], \alpha \in (0,2); \\ W_{rad}^{1,2}(\mathbb{R}^N, V) \hookrightarrow L^r(\mathbb{R}^N), & r \in [2^*, 2_\alpha^*], \alpha \in (2, 2N-2); \\ W_{\mathrm{rad}}^{1,2}(\mathbb{R}^N, V) \hookrightarrow L^r(\mathbb{R}^N), & r \in [2^*, \infty), \alpha \in [2N-2, \infty). \end{cases}$$

The embeddings are compact if $r \neq 2_\alpha^*$ and $r \neq 2^*$.

Obviously, Proposition B signifies that 2_α^* is the embedding bottom index and 2^* is the embedding top index when α belongs to $(0,2)$. However, for $\alpha \in (0,2)$, it gives that

$$2_\alpha^* = 2 + \frac{4\alpha}{2N-2-\alpha} < 2 + \frac{4\alpha}{2N-2-2} = \frac{2N-4+2\alpha}{N-2} =: \hat{2}_\alpha^*. \tag{1.3}$$

Hence, we believe that it is worth exploring the validity of the conclusions in the above Theorem A or Theorem 1.1 for the case $N \geq 3, \alpha \in (0,2)$ and $q \in \left(2_\alpha^*, \hat{2}_\alpha^*\right]$ in the future work.

2. Proof of Theorem 1.1

We firstly introduce the associated energy space for problem (1.1). For $N \geq 3$, denote by $L^p(\mathbb{R}^N)$, $p \in [1, \infty)$, the usual Lebesgue space with its norm $\|u\|_p := \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}$. Let $D^{1,2}(\mathbb{R}^N) := \{u \in$

$L^{2^*}(\mathbb{R}^N)$:$\int_{\mathbb{R}^N}|\nabla u|^2dx < \infty\}$ with the semi-norm $\|u\|_{D^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{1}{2}}$. Define $W^{1,2}(\mathbb{R}^N,V) := \{u \in D^{1,2}\mathbb{R}^N : \|u\|^2_{L^2(\mathbb{R}^N,V)} = \int_{\mathbb{R}^N}V(|x|)|u|^2dx < \infty\Big\}$ normed by

$$\|u\|^2_{W^{1,2}(\mathbb{R}^N,V)} := \left(\int_{\mathbb{R}^N}|\nabla u|^2dx + \int_{\mathbb{R}^N}V(|x|)|u|^2dx\right)^{\frac{1}{2}}$$

and $W^{1,2}_{\mathrm{rad}}(\mathbb{R}^N,V)$ is the set of radial functions in $W^{1,2}(\mathbb{R}^N,V)$. Note that the embeddings of $D^{1,2}(\mathbb{R}^N,V) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $W^{1,2}_{\mathrm{rad}}(\mathbb{R}^N,V) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ are continuous, it allows us to define the best embedding constants S and S_α:

$$S := \inf_{u\in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{\frac{2}{2^*}}} = \inf_{u\in D^{1,2}(\mathbb{R}^N)} \frac{\|\nabla u\|_2^2}{\|u\|^2_{2^*}} \tag{2.1}$$

and

$$S_\alpha := \inf_{u\in W^{1,2}_{rad}(\mathbb{R}^N,V)} \frac{\int_{\mathbb{R}^N}|\nabla u|^2dx + \int_{\mathbb{R}^N}V(|x|)|u|^2dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{\frac{2}{2^*}}} = \inf_{u\in W^{1,2}_{rad}(\mathbb{R}^N,V)} \frac{\|u\|^2_{W^{1,2}_{rad}(\mathbb{R}^N,V)}}{\|u\|^2_{2^*}} \tag{2.2}$$

Especially, S is achieved at

$$U_\sigma = [N(N-2)]^{\frac{(N-2)}{4}}\left(\frac{\sigma}{\sigma^2+|x|^2}\right)^{\frac{N-2}{2}} := A_N\left(\frac{\sigma}{\sigma^2+|x|^2}\right)^{\frac{N-2}{2}} \text{ for any given } \sigma > 0, \tag{2.3}$$

which satisfies $\|U_\sigma\|^2_{D^{1,2}(\mathbb{R}^N)} = \|U_\sigma\|^{2^*}_{2^*} = S^{\frac{N}{2}}$.

Equation (1.1) is variational and its solutions are the critical points of the functional defined in $W^{1,2}_{\mathrm{rad}}(\mathbb{R}^N,V)$ by

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}V(|x|)|u|^2dx - \frac{\lambda}{2^*_\alpha}\int_{\mathbb{R}^N}|u|^{2^*_\alpha}dx - \frac{1}{q}\int_{\mathbb{R}^N}|u|^q dx - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx.$$

According to [8, Lemma 5.1], we have known that the functional J possesses the mountain pass geometry under the assumptions of Theorem 1.1. Moreover, define $c := \inf_{u\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma(t))$ and $\bar{c} := \inf_{u\in\mathcal{N}} J(u)$, it also gives that $c = \bar{c} = \bar{\bar{c}} := \inf_{u\in W^{1,2}_{\mathrm{rad}}(\mathbb{R}^N,V)\setminus\{0\}}\sup_{t\in[0,1]} J(tu) > 0$, where $\Gamma := \{\gamma \in C\left([0,1], W^{1,2}_{\mathrm{rad}}(\mathbb{R}^N,V)\right) : \gamma(0) = 0, J(\gamma(1)) < 0\}$ and $\mathcal{N} := \{u \in W^{1,2}_{\mathrm{rad}}(\mathbb{R}^N,V)\setminus\{0\} : \langle J'(u),u\rangle = 0\}$. To obtain the existence of ground states of problem (1.1), the essential step is to verify that the mountain pass level c defined above belongs to some reasonable interval. For this purpose, we choose the following test functions U_n different from [8] and make subtle analyses. For the convenience of discussion, in what follows, N is fixed, that is, $N = 3$.

Lemma 2.1. *Under the assumptions of Theorem 1.1, one has* $0 < c < \frac{1}{3}S^{\frac{3}{2}}$.

Proof. Consider the functions $U_n(x) := H_n(|x|), n \in \mathbb{N}^+$, where

$$H_n(r) := A_3 \begin{cases} \left(\frac{n}{1+n^2r^2}\right)^{\frac{1}{2}}, & 0 \le r < 1; \\ \left(\frac{n}{1+n^2}\right)^{\frac{1}{2}}(2-r), & 1 \le r < 2; \\ 0, & r \ge 2, \end{cases}$$

where A_3 is determined in (2.3). By direct calculations, we have

$$\begin{aligned} \|U_n\|_2^2 &= \int_{\mathbb{R}^3} |U_n|^2 dx = \omega_3 \int_0^\infty r^2 |H_n(r)|^2 dr \\ &= \omega_3 A_3^2 \left(\int_0^1 \frac{nr^2}{1+n^2r^2} dr + \frac{n}{1+n^2} \int_1^2 r^2(2-r)^2 dr \right) \\ &= \omega_3 A_3^2 \left(\frac{1}{n^2} \int_0^n \frac{s^2}{1+s^2} ds + \frac{8}{15} \frac{n}{1+n^2} \right) \\ &= O\left(\frac{1}{n}\right) \end{aligned} \tag{2.4}$$

$$\begin{aligned} \left\|\nabla U_n\right\|_2^2 &= \int_{\mathbb{R}^3} \left|\nabla U_n\right|^2 dx = \omega_3 \int_0^{+\infty} r^2 \left|H_n'(r)\right|^2 dr \\ &= \omega_3 A_3^2 \left(\int_0^1 \frac{n^5 r^4}{(1+n^2r^2)^3} dr + \frac{n}{1+n^2} \int_1^2 r^2 dr \right) \\ &= \omega_3 A_3^2 \left(\int_0^n \frac{s^4}{(1+s^2)^3} ds + \frac{7}{3} \frac{n}{1+n^2} \right) \\ &= S^{\frac{3}{2}} + \omega_3 A_3^2 \left(-\int_n^{+\infty} \frac{s^4}{(1+s^2)^3} ds + \frac{7}{3} \frac{n}{1+n^2} \right) \\ &= S^{\frac{3}{2}} + O\left(\frac{1}{n}\right) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} |U_n|^{2^*} dx = \|U_n\|_{2^*}^{2^*} &= \omega_3 \int_0^\infty r^2 |H_n(r)|^{2^*} dr \\ &= \omega_3 A_3^{2^*} \left(\int_0^1 \frac{n^3 r^2}{(1+n^2r^2)^3} dr + \left(\frac{n}{1+n^2}\right)^3 \int_1^2 r^2 (2-r)^{2^*} dr \right) \\ &= \omega_3 A_3^{2^*} \left(\int_0^n \frac{s^2}{(1+s^2)^3} ds + \left(\frac{n}{1+n^2}\right)^3 \int_0^1 s^{2^*}(2-s)^2 ds \right) \\ &= S^{\frac{3}{2}} + O\left(\frac{1}{n^3}\right) \end{aligned} \tag{2.6}$$

where ω_3 is the area of the unit sphere in $\mathbb{R}^3$. Moreover, according to (V_3), we infer that

$$\begin{aligned} \int_{\mathbb{R}^3} V(|x|)|U_n|^2 dx &\le \int_{\mathbb{R}^3} \frac{B}{|x|^\alpha} |U_n|^2 dx \\ &= \omega_3 B A_3^2 \int_0^{+\infty} r^{2-\alpha} |H_n(r)|^2 dr \\ &= \omega_N B A_3^2 \left(\int_0^1 \frac{nr^{2-\alpha}}{1+n^2r^2} dr + \frac{n}{1+n^2} \int_1^2 r^{2-\alpha}(2-r)^2 dr \right) \end{aligned} \tag{2.7}$$

Take into account that

$$\omega_3 B A_3^2 \frac{n}{1+n^2} \int_1^2 r^{2-\alpha}(2-r)^2 dr = O\left(\frac{1}{n}\right)$$

and

$$\int_0^1 \frac{nr^{2-\alpha}}{1+n^2r^2}dr = n^{\alpha-2}\int_0^n \frac{s^{2-\alpha}}{1+s^2}ds < n^{\alpha-2}\int_0^n s^{-\alpha}ds = n^{\alpha-2}O(n^{1-\alpha}) = O\left(\frac{1}{n}\right)$$

it follows from (2.7) that

$$\int_{\mathbb{R}^N} V(|x|)|U_n|^2dx = O\left(\frac{1}{n}\right). \tag{2.8}$$

As far as $\|U_n\|_q^q$ is concerned, we have

$$\int_{\mathbb{R}^3} |U_n|^q dx = \omega_3\int_0^{+\infty} r^2|H_n(r)|^q dx \geq \omega_3 A_3^q \int_0^1 \frac{n^{\frac{q}{2}}r^2}{(1+n^2r^2)^{\frac{q}{2}}}dr$$

$$= \frac{\omega_3 A_3^q}{n^{\frac{6-q}{2}}}\int_0^n \frac{s^2}{(1+s^2)^{\frac{q}{2}}}ds \tag{2.9}$$

$$\geq \frac{\omega_3 A_3^q}{n^{\frac{6-q}{2}}}\int_0^1 \frac{s^2}{2^{\frac{q}{2}}}ds =: \frac{C}{n^{\frac{6-q}{2}}}$$

which is available for both $q = 2_\alpha^*$ and $q \in (4, 2^*)$.

In the sequel, for any fixed n, we consider the following map

$$J(tU_n) := \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla U_n|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} V(|x|)|U_n|^2dx - \lambda\frac{t_\alpha^2}{2_\alpha^*}\int_{\mathbb{R}^3} |U_n|^{2_\alpha^*}dx - \frac{t^q}{q}\int_{\mathbb{R}^3} |U_n|^q dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^3} |U_n|^{2^*}dx, t \geq 0$$

It is obvious that $J(tU_n) \to -\infty$ as $t \to +\infty$ and $J(tU_n) \to 0^+$ as $t \to 0^+$. Thus, we can assume the existence of $t_n > 0$ to guarantee that $J(t_nU_n)$ takes its maximum value at t_n and

$$\|\nabla U_n\|_2^2 + \int_{\mathbb{R}^3} V(|x|)|U_n|^2dx = \lambda t_n^{2_\alpha^*-2}\|U_n\|_{2_\alpha^*}^{2_\alpha^*} + t_n^{q-2}\|U_n\|_q^q + t_n^{2^*-2}\|U_n\|_{2^*}^{2^*}. \tag{2.10}$$

From (2.4), (2.5), (2.6), (2.8), (2.10) and

$$\|U_n\|_q^q \leq \|U_n\|_2^{\frac{2(2^*-q)}{2^*-2}}\|U_n\|_{2^*}^{\frac{2^*(q-2)}{2^*-2}}, \|U_n\|_{2_\alpha^*}^{2_\alpha^*} \leq \|U_n\|_2^{\frac{2(2^*-2^*)}{2^*-2}}\|U_n\|_{2^*}^{\frac{2^*(2_\alpha^*-2)}{2^*-2}},$$

we know that $\{t_n\}$ possesses the uniformly (positive) lower bound and upper bound. As a result, when $N = 3, \alpha \in (0,1)$ and $q \in (4, 2^*)$, together with (2.5), (2.6), (2.8) and (2.9), we infer that for n large enough

Theorem A.([8, Theorem 1.1]) Assume that $q \in \left(\frac{2N-4+2\alpha}{N-2}, 2^*\right)$ and conditions $(V_1) - (V_3)$ hold. Let

$$\begin{cases} N = 3, & \alpha \in (1,2); \\ N \geq 4, & \alpha \in (0,2), \end{cases}$$

and

$$\lambda^* := \left(\frac{\left(\frac{1}{2}-\frac{1}{2_\alpha^*}\right)S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-2}}}{\left(\frac{1}{2}-\frac{1}{2^*}\right)S_\alpha^{\frac{2^*}{2^*-2}}}\right) \geq \lambda > 0, \tag{1.2}$$

Proof of Theorem 1.1. After establishing Lemma 2.1, the remaining is just to repeat [8, Lemma 5.3] to finish the proof of our Theorem 1.1. So we omit the details.

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