Original Research Article

Rainbow Coloring of Random Graphs

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Abstract: This paper explores the issue of identifying rainbow embeddings within random graphs, which occur when all the edges of a subgraph are colored distinctly. Using probabilistic methods, we investigate the conditions under which a random graph contains such an embedding. Specifically, for a specified graph H, when p is greater than the threshold, randomly select a color from the set of colors c to color the edges of graph G, then with high probability graph G contains a rainbow copy of H. These results provide new insights into the interplay between random graph theory and edge coloring, with potential applications in areas such as network design and combinatorics.

Keywords: Random graphs; Rainbow coloring; Probability method

1. Introduction

Graph theory, with roots tracing back over 200 years to Euler's famous solution of the Königsberg bridge problem [6], has since become a fundamental mathematical discipline. It plays a crucial role across various domains, including computer science, biology, and network theory, where complex systems can often be represented as graphs. One problem that has garnered increasing interest is the study of random graphs and edge colorings, both for their theoretical significance and practical applications [1].

A particularly notable challenge in this area is the rainbow coloring problem, where the aim is to assign distinct colors to the edges of a graph so that no two edges within the same subgraph share the same color. When all edges in a subgraph are colored uniquely, this subgraph is referred to as a rainbow subgraph. The process of embedding a rainbow-colored subgraph within a larger graph, known as rainbow embedding, has become a central topic in random graph studies, with applications ranging from communication networks to distributed systems.

This paper investigates the existence of rainbow embeddings within random graphs, focusing on the probabilistic properties that determine when such embeddings occur. Building on earlier research in random graphs and combinatorics, we extend the analysis by identifying new conditions under which a random graph will almost certainly contain a rainbow copy of a specified subgraph. Utilizing probabilistic techniques, we derive tighter constraints on edge probabilities and the number of colors needed to ensure the presence of rainbow subgraphs. These results not only enhance the theoretical understanding of random graph dynamics but also offer insights relevant to real-world problems where edge coloring plays a key role.

The structure of the paper is as follows: Section 2 introduces the primary theoretical results, outlining the conditions required for rainbow embeddings in random graphs. In Section 3, we present supporting lemmas that assist in proving the main theorem. Section 4 contains a detailed proof of the theorem, followed by a discussion of its broader implications in Section 5.

2. Conditions for Rainbow Embeddings in Random Graph

In this section, we outline the main theoretical results of this paper, focusing on the conditions that ensure random graphs contain rainbow embeddings of specific subgraphs. We analyze the random graph model $\mathcal{G}(n, p)$, exploring how the number of colors, edge probability p, and the size of the subgraph *H* interact to ensure that a rainbow copy of *H* appears with high probability. Our results offer refined bounds compared to existing work and shed new light on the probabilistic properties governing rainbow colorings in random graphs. Below, we introduce the results achieved by previous researchers and the conclusions obtained from the improvements made in this paper.

Let *G* be a graph (*G* belongs to $\mathcal{G}(n, p)$) where each edge is uniformly colored at random using one of the colors from the colors set $[c] := \{1,...,c\}$. This model is known as $\mathcal{G}_c(n, p)$. For a specified graph *H*, if *G* includes a subgraph that is a copy of *H* with all edges colored in different colors, we state that a typical member of $G \sim \mathcal{G}_c(n, p)$ has a rainbow copy of *H*. In [3], Frieze and Loh proved that for $p \geq (1+\varepsilon)\log n / n$ and $c = n + o(n)$, a typical member of $\mathcal{G}_c(n, p)$ contains a rainbow Hamilton cycle. Observe that their results are asymptotically optimized in terms of both *p* and the number of colors *c*. We will show that the boundary of *p* (edge probability) in the following theorem, then for the edge probability *p*, for any graph *H* with *n* vertices and $\Delta(H) = O(1)$, we can find a rainbow copy of *H* in a typical member of $\mathcal{G}_c(n, p)$, as long as $c = (1+o(1)) | E(H) |$.

Theorem 1. Let *n* be a sufficiently large integer, let $\alpha > 0$, let $\Delta > 1$ and d_m be integers, let $H \in \mathcal{H}(n, \Delta, d_m)$. Then $G \sim \mathcal{G}_c(n, p)$ with high probability contains a rainbow copy of *H*, provided that $p \geq n^{-1/d_m} \log^{3/d_m} n$ and $c = (1+\alpha) | E(H) |$.

3. Auxiliary lemmas

This section outlines several auxiliary lemmas that form the foundation for the proof of the main theorem. We utilize probabilistic bounds, particularly the Chernoff bound, to control the deviation of random variables in our random graph model. These lemmas help establish the conditions under which the rainbow subgraph embeddings are guaranteed.

Prior to embarking on the proof, it would be advantageous to establish the subsequent notation for clarity and efficiency throughout the subsequent discourse. Given any bipartite graph $G = (A \bigcup B, E)$ and $|A| = |B| = n$ with a specified minimum degree $\delta(G) \geq k$, define a collection of bipartite graphs as $B^{\ell}_{k-out}(G)$ where each member $D \in B^{\ell}_{k-out}(G)$ within this set possesses the same vertex set $V(D) = V(G)$ and an edge set $E(D) \subseteq E$, constrained such that every vertex belonging to set A exhibits a precise degree of k. Notably, a uniform random selection of an element from $B^{\ell}_{k-out}(G)$ can be achieved by individually and randomly selecting *k* edges from $E_G(v, B)$ for each vertex $v \in A$.

We need to limit the large deviation of the random variable. We will primarily rely on the celebrated Chernoff inequality, which provides a tight bound on both the lower and upper extremities of the binomial distribution's tail probabilities (see [5]).

Lemma 1. If $X \sim Bin(n, p)$, then

$$Pr(X < (1-a)np) < e^{-a^2 np/2} \text{ for every } a > 0;$$

$$Pr(X > (1+a)np) < e^{-a^2 np/3} \text{ for every } 0 < a < 3/2.$$

Subsequently, we present more universally applicable bounds that are derived in a straightforward manner from the fundamental principles embodied in Chernoff's bound [2].

Lemma 2. Let $p, q \in [0,1]$ and let $X_1, ..., X_n \in \{0,1\}$ be *n* indicator variables and $X := \sum_{i=1}^{n} X_i$. If for each $1 \le i \le n$

$$\mathbb{E}[X_i \mid X_1, ..., X_{i-1}] \ge p \quad \text{and} \quad \mathbb{E}[X_i \mid X_1, ..., X_{i-1}] \le q,$$

it holds for all $0 < a < 1$ that

$$Pr(X \le (1-a)np) \le e^{-a^2 np/2} \quad \text{and} \quad Pr(X \ge (1+a)nq) < e^{-a^2 nq/3}.$$

It is obviously that all graph $H \in \mathcal{H}(n, \Delta, d_m)$ is d_m-degenerate (but not vice versa). The subsequent observation is an immediate consequence of the very definition of d_m-degenerate graphs, showcasing a direct link between the two concepts.

Observation 1. Let *n* be a positive integer, and let $\Delta > 1$, $d_m > 0$ be integers. Let *H* be a d_m-degenerate graph on n vertices. Then it can ascertain the existence of a particular ordering $(v_1, ..., v_n)$ of the vertices in graph H such that the given condition

$$|N(v_i) \bigcap \{v_1, ..., v_{i-1}\}| \le d_m$$

hold for every $2 \le i \le n$.

Lemma 3. Let G be a graph on n vertices with maximum degree $\Delta \ge 2$, let $D \subseteq V(G)$ and the maximum degree of all vertices in D is at most d_m (where $d_m \ge 1$). Then, D includes a set $T \subseteq D$ of size at least $\frac{|D|}{d_m \Delta^k}$ which is *k*-independent in *G*.

In the later proof of main theorem, the following succinct lemma unveils a fundamental truth about the existence of perfect matchings in typical graphs from $B_{k-out}^{\ell}(G)$ is one of the main ingredients.

Lemma 4. Let n be a sufficiently large integer, let $\varepsilon > 0$, $k = \omega(\log n)$. Then for any bipartite graph $G = (A \bigcup B, E)$ with $|A| = |B| = n$ and $\delta(G) \ge n/2 + \varepsilon n$, a graph *D* chosen uniformly at random from $B_{k-out}^{\ell}(G)$ with high probability contains a perfect matching.

Proof. Let D be a graph chosen uniformly at random from $B_{k-out}^{\ell}(G)$. We prove that with high probability all subsets $T \subset A$ and all subsets $T \subset B$ with $|T| \le n/2$ satisfy $|T| \le |N_D(T)|$. It then follows from Hall's theorem (readers could be referred to [7] for more details) that D has a perfect matching.

Initially, we suppose that $T \subset A$. Observe that $|T| > |N_D(T)|$ implies that there exists a subset $T' \subset B$ of size $|T'| = |T| - 1$ such that $|E_D(T, B \setminus T')| = 0$. Observe that in G, since $|T'| \le n/2$, every vertex $v \in T$ has at least εn neighbors in $B \setminus T'$. In selecting the i-th edge among the k edge incident to vertex v, we meticulously consider the condition that none of the edges in $E_G(v, B \setminus T')$ have been previously chosen, the probability to miss $B \setminus T'$ is at most

$$\frac{d_G(v)-\varepsilon n-i+1}{d_G(v)-i+1}\le 1-\varepsilon.$$

Therefore,

$$Pr[|T|>|N_D(T)|] \le Pr[\exists T'\subset B \,||\, E_D(T,B\setminus T')|=0]$$
$$\le \binom{n}{|T|-1}(1-\varepsilon)^{|T|k} \le e^{-\varepsilon|T|\cdot\omega(\log n)},$$

and the probability that bad condition exists is no more than

$$\sum_{t=1}^{n/2}\binom{n}{t}e^{-\varepsilon t\cdot\omega(\log n)} \le \sum_{t=1}^{n/2}\binom{n}{t}e^{-t\cdot\omega(\log n)} = o(1).$$

Next, suppose that $T\subset B$ so as to have $|T|>|N_D(T)|$, there should exist a set $T'\subset A$ of size $|T|-1$ such that $|E_D(A\setminus T',T)|=0$.

Observe that $|E_G(A,T)|\ge|T|\cdot(n/2+\varepsilon n)$, $|E_G(T',T)|\le|T|\cdot|T'|\le|T|\cdot n/2$,hence $|E_G(A\setminus T',T)|\ge|T|$ εn. Every edge in *G* has at least a k/n chance of appearing in *D*. Knowing that another edge is not in *D* can only make this probability lower, it follows that

$$Pr[|T|>N_D(T)] \le Pr[\exists T'\subset A \,||\, E_D(T,A\setminus T')|=0]$$
$$\le \binom{n}{|T|-1}\Big(1-\frac{\omega(\log n)}{n}\Big)^{|T|\varepsilon n} \le e^{-\varepsilon|T|\cdot\omega(\log n)}$$

as in the earlier scenario.

4. Proof of main theorem

This section is dedicated to proving the main theorem. Our proof is motivated by ideas of Cooper and Frieze [4], and the proof idea is the same as method in [2], except that the parameters have been improved and some tighter bounds have been obtained. Notably, the presence of a rainbow copy of a fixed graph H is a monotone increasing property. Consequently, we can confidently set the probability p to exactly $n^{-1/d_m}\log^{3/d_m} n$.

Initially, we build a 'good' partition, followed by outlining the procedure to locate the rainbow instance of *H*.

Let n be a sufficiently large integer, let $\alpha>0$ be some arbitrarily small positive constant, let $\Delta>1$ and d_m be integers, let $H\in\mathcal{H}(n,\Delta,d_m)$. Moreover, let $\bar{d}=2|E(H)|/n$ denote the average degree of H. Initially, we demonstrate how to partition *H* in a manner that facilitates obtaining a rainbow copy within a typical member of $\mathcal{G}_c(n,p)$, where $c=(1+\alpha)|E(H)|$. To achieve this, we proceed with the following steps. If H contains a set W of $\left\lceil\frac{\alpha n}{5\log^2 n}\right\rceil$ isolated vertices (that is, vertices of degree 0 in H), then partition $V(H)=\{w_1\}\cup\ldots\cup\{w_t\}\cup W$ in such a way that for each i, the vertex w_i has at most d_m neighbors in $\{w_1,\ldots,w_{i-1}\}$. In fact, since $H':=H-W\in\mathcal{H}(n-|W|,\Delta,d_m)$, Thus, this partition exists and is d_m -degenerate, allowing us to apply Observation 1. Otherwise, let x denote the number of vertices of degree larger than 0 and at most $\bar{d}$ in *H*. Since *H* contains at most $\frac{\alpha n}{5\log^2 n}$ isolated vertices, then the following inequality is valid:

$$\bar{d}n = 2|E(H)| \geq x + (\bar{d}+1)\left(n - \frac{\alpha n}{5\log^2 n} - x\right).$$

Therefore, we confirm that $x \geq n/(2\bar{d})$ by using the fact that *n* is sufficiently large. Now, let S be the set comprising all the vertices under consideration. By leveraging Lemma 3, specifically applied to H and S, we derive a crucial conclusion. There exists a subset $T \subseteq S$, that possesses two key properties: T is 2-independent and its size

$$|T| \geq \frac{|S|}{\bar{d}\Delta^2} \geq \frac{n}{2\bar{d}^2\Delta^2} \geq \left\lceil \frac{\alpha n}{5\log^2 n} \right\rceil$$

for sufficiently large values of *n*. Next, let $W \subseteq T$ be an arbitrary subset of size $\left\lceil \frac{\alpha n}{5\log^2 n} \right\rceil$, and partition $V(H) = \{w_1\} \cup \ldots \cup \{w_t\} \cup W$ in such a way that for each i, w_i has at most d_m neighbors in $\{w_1, \ldots, w_{i-1}\}$.

In a nutshell, we obtain a partition $V(H) = \{w_1\} \cup \ldots \cup \{w_t\} \cup W$ such that $|W| = \left\lceil \frac{\alpha n}{5\log^2 n} \right\rceil$ and one of the following holds:

1. All the vertices of *W* are isolated in *H*, or
2. *W* is 2-independent and consists of non-isolated vertices of degree at most $\bar{d}$.

Observe that if (2) holds then

$$|E(W, V \setminus W)| \leq \bar{d}\,|W| = \frac{2|E(H)|}{n} \cdot \left\lceil \frac{\alpha n}{5\log^2 n} \right\rceil < \frac{\alpha |E(H)|}{2\lceil \log^2 n \rceil},$$

for *n* large enough.

We begin by outlining the method for identifying the rainbow copy of H. Let $q \geq p/2$ be such that $1-p = (1-q)^2$ and present $G \sim \mathcal{G}(n,p)$ as $G = G_1 \cup G_2$, where G_1 and G_2 are two graphs sampled independently from $\mathcal{G}(n,q)$, obviously $q \leq p$. We sample a member of $\mathcal{G}_c(n,p)$ by sampling a member of $\mathcal{G}(n,p)$ and the exposed edges are randomly colored using c colors.

The process of embedding a rainbow copy of H in $G \sim \mathcal{G}_c(n,p)$ is done in two phases. In the first phase, we obtain a rainbow embedding f of $H[\{w_1 \cup \ldots \cup w_t\}]$ with edges selected from G_1. If W is as in (1) (that is to say, all the vertices in W are isolated in H), the embedding is complete. Otherwise, in the second phase, we show that it is possible to extend f into a full rainbow embedding *H* in *G*, using edges of G_2.

Next, we detail the strategies employed in Phases I and II, demonstrating that the process succeeds with high probability.

Phase I: Throughout Phase I, we maintain a partial rainbow embedding f of H to G_1, a set of available vertices V' and a set of available colors $\mathcal{C}$. Initially, set $f = \varnothing$, $\mathcal{C} := [c]$ and $V' := V(G)$. Additionally, for each vertex $v \in V(G)$ we maintain a set $U_v \subseteq V(G)$ such that $U_v \cap V'$ contains only unexposed potential neighbors of v in G_1. At the start, $U_v = V(G) \setminus \{v\}$ for each $v \in V(G)$.

We build the partial embedding *f* step-by-step. In the first step, let $f(w_1) := v$ for an arbitrary vertex

$v \in V'$, and set $V' := V' \setminus \{v\}$. Suppose that we have already embedded $\{w_1, ..., w_{i-1}\}$ for some $2 \le i \le t$ and we want to embed $w := w_i$. Let $L(w_i) = f(N_H(w_i) \cap \{w_1, ..., w_{i-1}\})$ be the set of images of neighbors of w_i that have already been embedded (recall that $|L(w_i)| \le d_m$). Let $A_w = V' \cap (\bigcap_{v \in L(w_i)} U_v)$ be the set of all available vertices that are still unexposed neighbors of all vertices in $L(w_i)$, and choose an arbitrary subset $S_w \subset A_w$ of size $s := \left\lceil \frac{\alpha n}{(4\Delta \log n)^2} \right\rceil$ (Lemma 5 shows that throughout Phase I this is indeed possible, in other words, A_w is of size at least s). Expose all edges between $L(w_i)$ and S_w, and assign uniformly at random colors to all the obtained edges. Let $x \in S_w$ be a vertex that is connected to all the vertices in $L(w_i)$ and such that all the colors assigned to edges $\{vx \mid v \in L(w_i)\}$ are distinct and belong to $\mathcal{C}$. This process is guaranteed by Lemma 5. We extend f by defining $f(w_i) := x$, update $U_v := U_v \setminus S_w$ for all $v \in L(w_i)$, $V' := V' \setminus \{x\}$ and

$$\mathcal{C} := \mathcal{C} \setminus \{col \in \mathcal{C} \mid \exists v \in L(w_i) \text{ such that } vx \text{ is colored in } col\}.$$

Then, if W is as in (1) (that is, all the vertices in W are isolated in H), then we are done. Otherwise, we continue to Phase II in the following.

Phase II: Let $V^* := V(G) \setminus f(V(H) \setminus W)$. Our purpose is to extend f with a valid embedding of W into V^*, using the edges of G_2, in such a way that the resulting embedding is a rainbow embedding.

For $w \in W$ let $L(w) := f(N_H(w))$ and let $\mathcal{L} = \{L(w) \mid w \in W\}$. Recall that W is 2-independent and hence for $u, v \in W$, we have $L(u) \cap L(v) = \varnothing$. Let $F = (\mathcal{L} \cup V^*, E_F)$ with edge set

$$E_F := \{Lv \mid L \in \mathcal{L}, v \in V^* \text{ and } \forall_{u \in L} uv \notin E(G_1)\}$$

be the base graph used to construct a bipartite auxiliary graph $B(\mathcal{L}, V^*)$. Edges that appear in G_1 are excluded because they cannot be recolored. Observe that $|\mathcal{L}| = |W| = |V^*|$ and that the conditions of Lemma 4 are satisfied by very rough estimation of F in the following.

Claim: It holds with high probability that $\delta(F) \ge \frac{4}{5}|V^*|$.

Proof. For every $L \in \mathcal{L}$ and $v \in V^*$ the edge $Lv \notin E_F$ iff there exists $u \in L$ for which $uv \in E(G_1)$. Since $G_1 \sim \mathcal{G}(n, q)$, we have that with high probability $\Delta(G_1) \le 2nq$, by applying Chernoff's bound. Moreover, since for every $L \in \mathcal{L}$ we get that $|L| \le \bar{d}$, it follows that $d_F(L) \ge |V^*| - \bar{d}2nq > 4|V^*|/5$. A similar argument prove that we have $d_F(v) \ge 4|V^*|/5$ for every $v \in V^*$.

Below, we describe a random process aimed at constructing a bipartite graph $\mathcal{B}(\mathcal{L}, W) \in \mathcal{B}^{\ell}_{\lceil \log^2 n \rceil - out}(F)$ by exposing edges from $G_2 \setminus G_1$ and randomly coloring them. Initially, let

$$\mathcal{C} := \mathcal{C} \setminus \{col \in [c] \mid \exists u, v \in E(H \setminus W) \; s.t. \{f(u), f(v)\} \text{ has color } col\}$$

observe that $|\mathcal{C}| \ge \alpha |E(H)|$, and choose an arbitrary ordering $L_1, ..., L_{|\mathcal{L}|}$ of the elements in $\mathcal{L}$. Then, in step $1 \le i \le |\mathcal{L}|$, set $N_i := N_F(L_i)$ and create $\lceil \log^2 n \rceil$ edges from L_i to vertices in N_i as the following: so long as $|N_{\mathcal{B}(\mathcal{L},V^*)}(L_i)|<\lceil \log^2 n \rceil$, iteratively select a vertex $v \in N_i$ uniformly at random, set $N_i := N_i \setminus \{v\}$ and expose all edges from v to vertices in L_i and color them uniformly at random with colors from [c] . Observe that the process will only fail at this stage if at some point $N_i = \varnothing$ while $|N_{\mathcal{B}(\mathcal{L},V^*)}(L_i)|<\lceil \log^2 n \rceil$. If all the edges are contained within G_2 and each edge has a distinct color, all chosen from the available set of colors $\mathcal{C}$, add $L_i v$ to $\mathcal{B}(\mathcal{L},V^*)$. In the end of step i, remove all the colors that were used on edges incident to L_i,

$$\mathcal{C} := \mathcal{C} \setminus \{col \in [c] \mid \exists L_i, \exists v \in N_{\mathcal{B}(\mathcal{L},V^*)}(L_i)\ s.t.\ uv \text{ has color } col\}.$$

If this process is successful, then every matching $\mathcal{M}$ in $\mathcal{B}(\mathcal{L},V^*)$ is obviously rainbow in the sense that all edges in

$$\{uv \mid \exists Lv \in \mathcal{M}\ s.t.\ u \in L\}$$

must have different colors that have not been used in the embedding during Phase I. Finally, if $\mathcal{B}(\mathcal{L},V^*)$ contains a perfect matching, then we are done.

In the following, we demonstrate that the processes of Phase I and Phase II are successful with high probability, and that the constructed graph $\mathcal{B}(\mathcal{L},V^*)$ contains a perfect matching.

Lemma 5. The process of Phase I is successful with high probability.

Proof. First, in the whole phase I, we claim that $|A_w| \geq \lceil \alpha n/(4\Delta \log n)^2 \rceil$ for every vertex $w \in V(H)$ which has not been embedded. The following four observations follow immediately from construction of embedding f

1. Observe that we have $U_v = V(G) \setminus \{v\}$ for each $v \in V(G)$ at the beginning of Phase I.
2. We update U_v only after embedding a vertex w for which $v \in L(w)$, then we delete the set S_w which is of size $s = \lceil \alpha n/(4\Delta \log n)^2 \rceil$ from U_v.
3. Every vertex v is a member of at most Δ sets $L(w)$ (recall that $\Delta(H) \leq \Delta$).
4. Observe that $|V'| \geq \lceil \alpha n / 5\log^2 n \rceil$ throughout Phase I (recall that we do not embed W in this phase).

Therefore, at any moment throughout Phase I, the following condition holds true, $|U_v \cap V'| \geq |V'| - 1 - \Delta \cdot \lceil \alpha n/(4\Delta \log n)^2 \rceil$, for each vertex $v \in V(G)$. Since $|L(w)| \leq \Delta$, we confirm that $|A_w| = |V' \cap (\bigcap_{v \in L(w)} U_v)| \geq |V'| - \Delta - \Delta^2 \cdot \lceil \alpha n/(4\Delta \log n)^2 \rceil \geq s$, for n large enough.

Next, we demonstrate that whenever we aim to embed a vertex w, there is at least one available candidate in V'.

Let $w \in V(H) \setminus W$, $X := |\{v \in S_w \mid L(w) \subseteq N_{G_1}(v)\}|$. Observe that X is the sum of i.i.d. indicator random variables X_v (for all $v \in S_w$) for which $X_v = 1$ iff $L(w) \subseteq N_{G_1}(v)$. Obviously, we have that $\mathbb{E}[X] \geq sq^{d_m} \geq \frac{\alpha n}{(4\Delta \log n)^2} \cdot \Omega(\frac{\log^3 n}{n}) = \Omega(\log n)$, then by applying Chernoff's bound, $Pr[X \leq \frac{\mathbb{E}[X]}{2}] = e^{-\Omega(\log n)} = o(\frac{1}{n})$.

Now, observe that $|\mathcal{C}| \geq \alpha |E(H)|$ throughout Phase I. Hence, the probability that for a vertex $x \in S_w$ with $L(w) \subseteq N_{G_1}(x)$, and all the edges to $L(w)$ have distinct colors from *C* is at least

$$\frac{\binom{\mathcal{C}}{\ell}}{((1+\alpha)|E(H)|)^{\ell}} \geq \Big(\frac{\alpha |E(H)|}{(1+\alpha)|E(H)|\ell}\Big)^{\ell} \geq \Big(\frac{\alpha}{(1+\alpha)d}\Big)^{d_m} =: \gamma > 0,$$

where $|L(w)| = \ell$. Thus, if $X \geq \mathbb{E}[X]/2$ then the probability that there is no such x is at most $(1-\gamma)^{|X|} \leq e^{-\gamma|X|} = e^{-\Omega(\log n)} = o(1/n)$.

Therefore, when we manage to embed *w* there exists with probability $1 - o(1/n)$ a vertex $x \in S_w$ for that the following holds: I. *x* is connected to all the vertices in $L(w)$. II. All of the colors assigned to the edges $\{\{v, x\} : v \in L(w)\}$ are distinct and belong to *C*.

Finally, observe that since we embed at most *n* vertices, we get that for every vertex w_i there exists a 'good' vertex $x \in S_w$ by applying the union bound. Then, the proof is completed.

Lemma 6. The process of Phase II is successful with high probability.

Proof. First, observe that the process can only fail when we have that $N_i = \varnothing$ and $|N_{\mathcal{B}(\mathcal{L},V^*)}(L_i)| < \lceil \log^2 n \rceil$ in some steps $1 \leq i \leq |\mathcal{L}|$. Hence, it suffices to prove that the process creates with probability $1 - o(1/n)$ the $\lceil \log^2 n \rceil$ required edges in an arbitrary fixed step $1 \leq i \leq |\mathcal{L}|$. Let $X_i := \{v \in N_F(L_i) \mid L_i \subseteq N_{G_2}(v)|\}$. Observe that $|X_i|$ is the sum of i.i.d. indicator random variables $X_{i,v}$ (for all $v \in N_F(L_i)$) for which $X_{i,v} = 1$ iff $L_i \subseteq N_{G_2}(v)$. Obviously, we have that (recall that $|L_i| \leq \overline{d} \leq d_m$)

$$\mathbb{E}[|X_i|] \geq |N_F(L_i)| q^{d_m} \geq \delta(F) \cdot \Omega\Big(\frac{\log^3 n}{n}\Big) \geq \frac{3}{4} \cdot \frac{\alpha n}{5\log^2 n} \cdot \Omega\Big(\frac{\log^3 n}{n}\Big) = \Omega(\log n).$$

We get that $Pr[|X_i| \leq \frac{\mathbb{E}[|X_i|]}{2}] = e^{-\Omega(\log n)} = o(1/n)$ by applying Chernoff's bound.

Next, let $Y_i := \{v \in X_i\}$, the vertex v satisfies the condition that all edges in $E(L_i, v)$ have distinct colors (color select from $\mathcal{C}$). Observe that $|Y_i|$ is the sum of i.i.d. indicator variables $Y_{i,v}$ (for all $v \in X_i$) for which $Y_{i,v} = 1$ if and only if all edges in $E(L_i, v)$ have different colors from $\mathcal{C}$. Since we have that $|E(W, V \setminus W)| \leq \alpha |E(H)| / (2\lceil \log^2 n \rceil)$ and we remove for each edge in $E(W, V \setminus W)$ at most $\lceil \log^2 n \rceil$ colors from $\mathcal{C}$, the number of available colors in $\mathcal{C}$ is invariably at least $\alpha |E(H)|/2$. Therefore, the probability that for a vertex $v \in X_i$ all the edges to L_i have distinct colors from $\mathcal{C}$ is at least

$$p_i = \frac{\binom{\mathcal{C}}{\ell}}{((1+\alpha)|E(H)|)^{\ell}} \geq \Big(\frac{\alpha |E(H)|/2}{(1+\alpha)|E(H)|\ell}\Big)^{\ell} \geq \Big(\frac{\alpha}{(1+\alpha)2\overline{d}}\Big)^{\overline{d}} =: \gamma > 0,$$

where $|L_i| = \ell \leq \overline{d}$, this lower bound for p_i remains valid regardless of all other color assignments made in earlier steps. Hence, if $|X_i| \geq \mathbb{E}[|X_i|]/2$, then the expectation of $|Y_i|$ is at least

$$\mathbb{E}[|Y_i|] \geq |X_i| \cdot \gamma = \Omega(\mathbb{E}[|X_i|]) = \Omega(\log n)$$

and by applying Chernoff's bound, we obtain that

$$Pr[|Y_i| < \frac{\mathbb{E}[|Y_i|]}{2} || X_i| \geq \frac{\mathbb{E}[|X_i|]}{2}] = e^{-\Omega(\log n)} = o(1/n) .$$

We confirm that the probability that our process fails is at most

$$\sum_{i=1}^{|\mathcal{L}|} Pr[|Y_i| \leq \lceil \log^2 n \rceil] \leq |\mathcal{L}| \cdot o(1/n) = o(1).$$

Ultimately, since we select a random ordering of the neighbors of L_i, every $\lceil \log^2 n \rceil$-tuple of neighbors of L_i has an equal probability of being included in $\mathcal{B}(\mathcal{L}, V^*)$, ensuring that the process samples an element of $\mathcal{B}^{\ell}_{\log^2 n-out}(F)$ uniformly at random.

Therefore, the processes of Phase I and Phase II are both successful by above lemmas. Then the proof of main theorem is completed.

5. Conclusion and Discussion

In this paper, we explored the conditions for the existence of rainbow embeddings in random graphs, using probabilistic techniques. By extending the random graph model $\mathcal{G}(n, p)$, we established that for certain edge probabilities and a sufficient number of colors, a random graph will almost certainly contain a rainbow copy of a given subgraph H. Our work enhances previous studies by tightening the bounds on both the edge probability p and the number of required colors, making the findings relevant for practical applications in areas like network design and distributed systems.

Our findings suggest several paths for future research. Extending this analysis to dynamic random graphs may provide more adaptable models for real-world networks. Refining the bounds on p and the number of colors could lead to more efficient algorithms for rainbow embeddings. Moreover, studying the relationship between rainbow embeddings and graph properties such as connectivity may yield further theoretical insights.

Despite these contributions, challenges remain. For example, we assume uniform edge coloring, which is not always applicable in practice. Adapting our methods for biased or non-uniform colorings would be an important future direction. While our results improve existing bounds, further refinement could enhance their practical relevance. Overall, our work deepens the theoretical understanding of rainbow embeddings and offers potential for both theoretical advances and real-world applications in graph theory.

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