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RESEARCH ARTICLE

New iterative methods for solving general harmonic-like variational inequalities

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ABSTRACT

In this paper, we introduce and study some new classes of general harmonic-like variational inequalities. Auxiliary principle technique is applied to propose some iterative schemes for computing the approximate solution. Convergence analysis is investigated under certain weaker conditions. Several new problems such as harmonic-like complementarity problems, representation theorems and related optimization problems are discussed. The techniques and ideas developed in this paper can be used to develop new techniques for these problems.

Keywords: Harmonic functions; harmonic Variational inequalities; auxiliary principle; iterative methods; convergence

1. Introduction

Convexity theory contains a wealth of novel ideas and innovative techniques, which have played the significant role in the development of almost all the branches of pure and applied sciences such as fixed point, variational inequalities and optimizations. It is well known fact that the minimum of the differentiable convex function on the convex set in anormed spaces can be characterized by an inequal- ity, which is called the variational inequality. Lions and Stampacchia [17] considered and studied the variational inequalities. They also emphasized that the Riesz-Frechet representation theorem and Lax- Milgram lemma are special cases of the variational inequalities. Variational inequality theory can be viewed as a novel extension and generalization of the variational principles. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry, mathematical finance, ma- chine learning, artificial intelligence and related areas. It is well known facts that variational theory provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization. It is amazing that variational inequalities have influenced various areas of pure and applied sciences and are still continue to influence the recent research, see [3–5,10,12–18,20–36].

Anderson et al. [2] have investigated several aspects of the harmonic convex sets and harmonic convex

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functions, which can viewed as important generalizations of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory. It is known that the total resis- tance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if ζ_1 and ζ_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula $\left(\frac{1}{\zeta_1}+\frac{1}{\zeta_2}\right)^{-1}=\frac{\zeta_1\zeta_2}{\zeta_1+\zeta_2}$, which is half the harmonic means. The Asian options with harmonic average have been considered in Al-Azemi et al. [1], which can be viewed as a new direction in the study of the risk analysis stock exchange and financial mathematics. The harmonic mean are being used to suggest some iterative methods for solving nonlinear equations. Noor et al [30] have shown that the minimum of the dif- ferentiable harmonic convex functions on the harmonic convex set can be characterized by a class of harmonic variational inequalities. Noor et al [28] introduced the new concepts of harmonic-like convex sets and harmonic-like convex functions. 2000 Mathematics Subject Classification. 41A05, 47A12, 49J40, 65K15, 90C33.

Motivated and inspired by this new direction, we introduce and consider some new classes of harmonic-like variational inequalities involving two arbitrary operators. Several new problems such as inverse harmonic-like, variational inequalities, inverse variational inequalities, harmonic-like com- plementarity problems, harmonic-like Lax-Milgram lemma are highlighted. Due to the structure and nonlinearity of the harmonic-like variational inequalities, the projection, resolvent operators and their variants cannot be applied to compute the approximate the solution. To overcome these drawback, one uses the auxiliary principle technique, which is mainly due to Lions and Stampachia [17] and Glowinski et al [11], to discuss various aspects of the such nonlinear problems. It have been shown in [9,20,28,29,32,34] that the auxiliary principle technique can be used effectively to suggest some iterative methods for solving various classes of variational inequalities and related optimization prob- lems. In Section 2, we introduce the general harmonic-like variational inequalities and discuss some important special cases. The auxiliary principle technique is applied to suggest some iterative methods for computing the approximate solution of harmonic-like variational inequalities. . Convergence anal- ysis of the proposed method is also considered under some mild conditions. We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate their numerical efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may can be extended for other classes of variational inequalities and related optimization problems.

2. Basic definations and results

Let Ω be a closed set in a real Hilbert space 孔 with norm II·II and inner product $\langle\cdot,\cdot\rangle$, respectively. Let 丁, g : 孔 ⟶ 孔 be a nonlinear operators. We now introduce some new concepts in harmonic convex analysis, which are mainly due to Noor et al [28].

Definition 2.1. [28] A set $\Omega \leq$ 孔 is said to be a general harmonic-like convex set with respect to an arbitrary operator g, if

$$\frac{2w\mu}{\mu + w} + t(\nu - g(\mu)) \in \Omega, \quad \forall \mu, \nu, w \in \Omega, t \in [0,1].$$

Note that, fort = 0, $\frac{2w\mu}{\mu + w} \in \Omega$, t = 1, $\left(\frac{2w\mu}{\mu + w} + \nu - g(\mu)\right) \in \Omega.$

Remarks. We now mention some main features of the harmonic-like convex sets for the sake of completeness and to convey the main ideas.

(1) For $w = \mu$, the general harmonize-like set Ω becomes a general convex set $\mu + t(\nu - g(\mu)) \in \Omega$, $\forall \mu, \nu, w \in \Omega, t \in [0, 1]$.

which was introduced by Noor [25] in 2008. Cristescu et al. [7–9] have discussed the several applications of general convex set in railway systems, information technology, computer aided design and digital vector optimization.

(2) If $w = \mu$ and $g = I$, identity operator, then the general harmonic-like conve set reduces to the classical convex set, see the excellent books [6,19] for more details.

(3) For $w = \nu$, the general harmonic-like convex set reduces to:

$$\frac{2\nu\mu}{\mu + \nu} + t(\nu - g(\mu)) \in \Omega, \quad \forall \mu, \nu \in \Omega, t \in [0,1],$$

which is called the mean harmonic convex set and appears to be a new.

Definition 2.2. [28] A function φ on the harmonic-like convex set Ω is said to be general harmonic- like convex function involving an arbitrary operator g, if

$$\varphi\left(\frac{2w\mu}{\mu + w}\right) + t(\nu - g(\mu)) \leq \varphi(g(\mu)) + t(\varphi(\nu) - \varphi(g(\mu))), \quad \forall \mu, \nu, w \in \Omega, \ t \in [0, 1].$$

3. General harmonic-like variational inequalities

We remark that, if $\mu = w$, them the general harmonic-like convex set becomes the general convex set and the general harmonic-like convex functions reduces to the general convex function. For the properties and applications of general convex sets and general convex functions, see [4,7–9,25,27,28,30] and the references therein

Using the technique of Noor et al [28], one can easily prove that the minimum of the differentiable general harmonic-like convex function can be characterized by an inequality.

Theorem 2.1. Let φ be a differentiable general harmonic-like convex function. Then $\mu \in \Omega$ is the minimum of the general harmonic-like convex function, if and only if, $\mu \in \Omega$ satisfies

$$\langle \varphi'(\frac{2w\mu}{\mu + w}), \nu - g(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.1}$$

which is called the general harmonic-like variational inequality.

In many cases, the inequalities of the type (2.1) may not arise as the optimality condition of the differentiable general harmonic-like convex functions. This motivated us to consider more general case, which include the problem (2.1) as a special case. This is the main motivation of this paper.

To be more precise, for given nonlinear operators $\mathcal{T}, g : \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), \nu - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.2}$$

which is called the general harmonic-like variational inequality.

Special Cases.

We now discuss some special cases of general harmonic-like variational inequalities (2.2).

(1) It is obvious that the problem (2.1) is a special case of the problem (2.2) with $\mathcal{T}(\frac{2w\mu}{w+\mu}) = \varphi'(\frac{2w\mu}{\mu+w})$.

(2) By interchanging the role of the operators $\mathcal{T}$ and g, the problem (2.2 is equivalent to finding $\mu \in \Omega$, such that

$$\langle g(\frac{2w\mu}{w+\mu}), \nu - \mathcal{T}(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.3}$$

which is also called the general harmonic-like variational inequality.

(3) For $\mathcal{T} = I$, the identity operator, then the problem (2.2) of finding $\mu \in \Omega$, such that

$$\langle (\frac{2w\mu}{w+\mu}), \nu - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.4}$$

which is called the inverse harmonic-like variational inequality and appears to be new one.

(4) If $w = \mu$, the problem (2.4) reduces to the problem of finding $\mu \in \Omega$, such that

$$\langle \mu, \nu - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.5}$$

which is called the inverse variational inequality. For the applications, motivation, general- izations and some aspects of the inverse variational inequalities, see [3,4,10,12–14,27,34,35] and the references therein.

(5) For $w = \mu$, the problem (2.2) collapses to finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}\mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.6}$$

is called the general variational inequality, introduced and studied by Noor [21] in 1988. For the applications and generalizations of the general variational inequalities, see [3,4,10,21,25, 27,30] and the references therein.

(6) If $w = \mu, g = I$, then the problem (2.2) reduces to finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}\mu, \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.7}$$

is called the variational inequality, introduced by Lions and Stampacchia [17]. It has been shown a wide class of obstacle boundary value and initial value problems can be studied in the general framework of variational inequalities. For the generalization, extensions, applications, motivation, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of variational inequalities, see [3,4,10–15,17,20–32,35,36] and the references therein.

(7) If $\Omega^* = \{\mu \in H : \langle \mu, \nu\rangle \geq 0, \forall \nu \in \Omega, \}$ is a polar(dual) cone, then the problem (2.2)is equivalent to finding $\mu \in \Omega$ such that

$$g(\mu) \in \Omega, \quad T\left(\frac{2w\mu}{w+\mu}\right) \in \Omega^*, \quad \left\langle T\left(\frac{2w\mu}{w+\mu}\right), g(\mu)\right\rangle = 0, \tag{2.8}$$

which is called the harmonic-like complementarity problem and appears to be a new one.

(8) If $\mu = w$, then the problem (2.8) reduces to finding $\mu \in \Omega$ such that

$$g(\mu) \in \Omega, \quad T\mu \in \Omega^*, \quad \langle T\mu, g(\mu)\rangle = 0, \tag{2.9}$$

which is called the general complementarity problem, introduced and studied by Noor [20]in 1988.

For the applications, motivations, generalization, numerical methods and other aspects of the complementarity problems in engineering and applied sciences, see [5,20,24,29,32] and the references therein.

(9) If $\Omega = H$, then the problem (2.2) collapses to finding $\mu \in H$ such that

$$\left\langle T\left(\frac{2w\mu}{w+\mu}\right), \nu - g(\mu)\right\rangle = 0, \quad \forall \nu \in H. \tag{2.10}$$

which is called the general harmonic-like Lax-Milgram Lemma, introduced and studied in [28].

(10) If $w = \mu$, then problem (2.10) reduces to finding $\mu \in \Omega$ such that

$$\langle T\mu, \nu - g(\mu)\rangle = 0, \quad \forall \nu \in H. \tag{2.11}$$

is known as the general Lax-Milgram lemma. For the applications, generalizations and ex- tensions of the Lax-Milgram Lemma, see [16,17,24,28,30] and the references therein.

For a different and appropriate choice of the operators and spaces, one can obtain several known and new classes of variational inequalities and related problems. This clearly shows that the problem (2.2) considered in this paper is more general and unifying one.

4. Main results

In this section, we use the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [17] and Glowinski et. al. [11], as developed in [22,24,28,30–32] to consider some approximate schemes for solving the general harmonic-like variational inequalities. The main of idea of this technique is to consider an arbitrary auxiliary problem related to the original problem. This way, one defines a mapping connecting the both problems. The novel feature of this approach is that this technique enables us to suggest some iterative methods for solving the harmonic-like variational inequalities and related optimization problems

We now use the auxiliary principle technique to suggest some iterative methods for computing the approximate solution of (2.2).

To be more precise, for a given $\mu \in \Omega$ satisfying (2.2), consider the problem of finding $\eta \in \Omega$ such that

$$\left\langle \rho T\left(\frac{2\mu w}{\mu+w}\right) + g(\eta) - g(\mu), \nu - g(\eta)\right\rangle \geq 0, \quad \forall \nu, w \in \Omega, \tag{3.1}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. It is clear that (3.1) defines a mapping η connecting the both problems (2.2) and (3.1).

5. General harmonic-like variational inequalities

It is worth mentioning that $\eta = \mu$ is a solution of (2.2). This implies that the auxiliary principle technique enables us to suggest the following iterative method for solving the problem (2.2).

Algorithm 3.1. For a given initial value μ_0, compute the approximate solution μ_{n+1} by the iterative scheme

$$\left\langle \rho\mathcal{T}\left(\frac{2\mu_n w}{\mu_n + w}\right) + g(\mu_{n+1}) - g(\mu_n), \nu - g(\mu_{n+1})\right\rangle \geq 0, \quad \forall \nu, w \in \Omega.$$

We again use the auxiliary principle technique to suggest an implicit method for solving the problem (2.2).

For a given $\mu \in \Omega$ satisfying (2.2), consider the problem of finding $\eta \in \Omega$ such that,

$$\left\langle \rho T\left(\frac{2w\eta}{w+\eta}\right) + g(\eta) - g(\mu), \nu - g(\eta)\right\rangle \geq 0, \quad \forall \nu, w \in \Omega, \tag{3.2}$$

which is called the auxiliary problem. We note that the auxiliary problems (3.1) and (3.2) are quite different. Clearly $\eta = \mu \in \Omega$ is a solution of (2.2). This observation allows us to suggest the following iterative method for solving the problem (2.2).

Algorithm 3.2. For a given initial value μ_0, compute the approximate solution μ_{n+1} by the iterative scheme

$$\left\langle \rho T\left(\frac{2u_{n+1} w}{\mu_{n+1} + w}\right) + g(\mu_{n+1}) - g(\mu_n), \nu - g(\mu_{n+1})\right\rangle \geq 0, \quad \forall \nu, w \in \Omega, \tag{3.3}$$

which is an implicit method, To implement this method, we use the predictor-corrector approach.. Consequently, we obtain a two-step iterative method for solving the problem (2.2).

Algorithm 3.3. For a given initial value $\mu_0 \in \Omega$, compute the approximate solution $\mu_{n+1} \in \Omega$ by the iterative scheme

$$\left\langle \rho T\left(\frac{2\mu_n w}{\mu_n + w}\right) + g(y_n) - g(\mu_n), (\nu - g(y_n)\right\rangle \geq 0, \quad \forall \nu, w \in \Omega.$$

$$\left\langle \rho T\left(\frac{2y_n w}{y_n + w}\right) + g(\mu_{n+1}) - g(u_n), \nu - g(u_{n+1})\right\rangle \geq 0, \quad \forall \nu, w \in \Omega,$$

which is known as two-step iterative method for solving problem (2.2). For the convergence criteria, we need the following concept.

Definition 3.1. An operator $T(\cdot)$ is said to be pseudo harmonic-like with respect to the operator g, if

$$\left\langle \mathcal{T}\left(\frac{2\mu w}{\mu + w}\right), \nu - g(\mu)\right\rangle \geq 0, \quad \forall \nu, w \in \Omega,$$
$$\Rightarrow \left\langle \mathcal{T}\left(\frac{2\nu w}{\mu + \nu}\right), g(\nu) - \mu\right\rangle \geq 0 \quad \forall \nu, w \in \Omega.$$

We now consider the convergence analysis of Algorithm 3.2 and this is the main motivation of our next result.

Theorem 3.1. Let $\mu \in \Omega$ be a solution of (2.2) and let μ_{n+1} be the approximate solution obtained from Algorithm 3.3. If the operator $T(\cdot)$ is pseudo harmonic-like operator, then

$$\|g(\mu_{n+1}) - \mu\|^2 \leq \|g(\mu_n) - \mu\|^2 - \|g(\mu_{n+1}) - g(\mu_n)\|^2. \tag{3.4}$$

Proof. Let $\mu \in \Omega$ be a solution of (2.2). Then

$$\left\langle T\left(\frac{2\nu w}{\cdot}\right), g(\nu) - \mu \right\rangle \geq 0, \quad \forall \nu, w \in \Omega, \tag{3.5}$$

since the operator T is pseudo harmonic-like monotone with respect to the operator g. Taking $\nu = \mu_{n+1}$ in (3.5) and $\nu = \mu$ in (3.3), respectively, we have

$$-\left\langle T\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right), \mu - g(\mu_{n+1}) \right\rangle \geq 0 \tag{3.6}$$

$$\left\langle \mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right) + g(\mu_{n+1}) - g(\mu_n), \mu - g(\mu_{n+1}) \right\rangle \geq 0. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$g(\mu_{n+1}) - g(\mu_n), \mu - g(\mu_{n+1}) \Big\rangle \geq -\langle \mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}, \mu - g(\mu_{n+1}) \geq 0,\right.$$

which implies that

$$\langle g(\mu_{n+1}) - g(\mu_n), \mu - g(\mu_{n+1}) \rangle \geq 0.$$

Using the result,

$$\forall a,b \in H, \quad 2\langle a,b \rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2,$$

we have

$$\|g(\mu_{n+1}) - \mu\|^2 \leq \|g(\mu_n) - \mu\|^2 - \|g(\mu_{n+1}) - g(\mu_n)\|^2,$$

the required (3.4). □

Theorem 3.2. Let $\mu \in \Omega$ be a solution of (2.2) and let μ_{n+1} be the approximate solution obtained from Algorithm 3.1. If all the assumptions of Theorem 3.1 holds and g^{-1} exists, then

$$\lim_{n\to\infty} \mu_{n+1} = \mu. \tag{3.8}$$

Proof. Let $\mu \in \Omega$ be a solution of (2.2). From (3.4), it follows that the sequence $\{\|g(\mu) - g(\mu_n)\|\}$ is noncreasing and consequently the sequence $\{\mu_n\}$ is bounded. Also, from (3.4), we have

$$\sum_{n=1}^{\infty} |g(\mu_{n+1}) - g(\mu_n)\|^2 \leq \|g(\mu_0) - \mu\|^2$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0, \tag{3.9}$$

since g^{-1} exists. Let $\hat{\mu}$ be a cluster point of $\{\mu_n\}$ and the subsequences $\{\mu_{n_j}\}$ of the sequence $\{\mu_n\}$ converge to $\hat{\mu}$. Replacing μ_n by $\{\mu_{n_j}\}$ in (3.3), taking the limit as $\lim_{n\to\infty} n_j \to \infty$ and using (3.9), we

have

$$\left\langle \mathcal{T}\left(\frac{2w\hat{\mu}}{v+\hat{\mu}}\right), \nu - g(\hat{\mu}) \right\rangle \geq 0, \forall \nu, w \in \Omega,$$

which shows that $\hat{\mu} \in H$ satisfies (2.2) and

$$\|g(\mu_{n+1}) - g(\mu_n)\|^2 \leq \|g(\mu_n) - \hat{\mu}\|^2.$$

From the above inequality, it follows that the sequence $\{\mu_n\}$ has exactly one cluster point $\hat{\mu}$ and $\lim_{n\to\infty} \mu_n = \hat{\mu}.$ □

We can apply the modified auxiliary principle technique involving an arbitrary operator, which is mainly due to Noor [26], to suggest hybrid iterative methods for solving the problem (2.2).

For a given $\mu \in \Omega$ satisfying (2.2), consider the problem of finding $\eta \in \Omega$ such that

$$\left\langle \rho \mathcal{T}\left(\frac{2\eta w}{\eta + w}\right) + M(g(\eta)) - M(g(\mu)) + \zeta(g(\mu) - g(\mu)), \nu - g(\eta)\right\rangle \geq 0, \quad \forall \nu, w \in \Omega, \tag{3.10}$$

where M is a nonlinear arbitrary operator and $\zeta \geq 0$ is an arbitrary parameter. For the excellent discussion and applications of the auxiliary principle techniques, see Patrikson [34] and the references therein.

For M = I the identity operator and $\zeta = 0$ the auxiliary problem is exactly the auxiliary problem (3.2). For suitable choice of the operator M and the parameter, one can obtain a wide class of auxiliary problems associated with problem (2.2). It is obvious that $\eta = \mu \in \Omega$ is a solution of the problem (2.2). This observation is used to suggest the general hybrid iterative methods for solving the problem (2.2) , which contain some new inertial iterative methods.

6. General harmonic-like variational inequalities

Algorithm 3.4. Forgiven $\mu_0, \mu_1 \in \Omega$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\left\langle \rho \mathcal{T}\left(\frac{2\mu_{n+1} w}{\mu_{n+1} + w}\right) + M(g(\mu_{n+1})) - M(g(\mu_n)) + \zeta(g(\mu_n) - g(\mu_{n-1})), \nu - g(\mu_{n+1})\right\rangle \geq 0, \quad \forall \nu, w \in \Omega.$$

Algorithm 3.4 is called the hybrid inertial implicit iterative method, which contains Algorithm 3.2 and the following inertial iterative method for M = 0.

Algorithm 3.5. Forgiven $u_0, u_1 \in$ 孔, compute the approximate solution u_{n+1} by the iterative scheme

$$\left\langle \rho \mathcal{T}\left(\frac{2\mu_{n+1} w}{\mu_{n+1} + w}\right) + \eta(g(\mu_n) - g(\mu_{n-1})), \nu - g(\mu_{n+1})\right\rangle \geq 0, \quad \forall \nu, w \in \Omega.$$

Algorithm 3.5 appears to be a new one. Using the predictor-corrector technique and updating the technique [24,28], one can propose and suggest some multi-step iterative schemes for solving the general harmonic-like variational inequalities and their variant forms. Using the technique of Noor et al. [28, 30], one can study the convergence criteria of Algorithm 3.5. For the applications and convergence analysis of the inertial type methods.

7. Conclusion

In this paper, we have introduced and analyzed some new classes of general harmonic- like variational inequalities. Several special cases are considered as applications of these new concepts. The auxiliary principle approach is applied to analyze some the approximate solutions of the general harmonic-like variational inequalities. Convergence analysis is considered under some weaker conditions, which do not require the strongly monotonicity and Lipschitz continuity of the involved operator. These results continue to hold for special cases such as harmonic-like complementarity problems, Lax-Milgram lemma and representation theorems. Our methods of investigation are very simple as compared with other methods. For the extensions, generalizations, modifications and applications of Noor Three-step) iterations [26] in various areas of pure and applied sciences such as design of solar energy panels [18], quantum calculus, logistic maps [37], information, AI, machine learning, data anal- ysis [36], stock exchange [1], financial mathematics [1], fuzzy set theory, fractal, fractional, random, stochastic process [34], computer design [7,8], optimization and operations research, see [34–36]. It is an open interesting problem to develop efficient numerical methods for solving the general harmonic- like variational inequalities using the Noor iterations. The theory of general harmonic-like variational inequalities does not appear to have developed to an extent that it provides a complete framework for studying these problems. Much more research is needed in all these areas to develop a sound basis for applications. It is interesting open problem to compare these techniques with other methods.

Contributions of the authors

All the authors contributed equally in writing, editing, reviewing and agreed for the final version for publication.

Data availability

Data sharing not applicable to this article as no data sets were generated.

Conflict of Interest

All authors have no conflict of interest.

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